Math 524 Exam 1 Solutions

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1. (15 pts)
   (a) Let $\circ$ be a binary operation on the set S. Define what it means for $\circ$ to be commutative.

   $\circ$ is commutative if $x \circ y = y \circ x$ for all $x, y \in S$.

   (b) Let $\circ$ be a binary operation on the set S. Define what it means for $\circ$ to be associative.

   $\circ$ is associative if $(x \circ y) \circ z = y \circ (x \circ z)$ for all $x, y, z \in S$.

   (c) Does an operation that is commutative also have to be associative? Make sure you carefully justify your decision.

   No. The operation in problem 2 is commutative but not associative.

2. (20 pts) Let $x \circ y = |x - y|$ on $\mathbb{R}^{\geq 0}$ (the nonnegative real numbers).
   (a) Is $\circ$ commutative?

   $$x \circ y = |x - y| = |y - x| = y \circ x$$

   for all $x, y \in \mathbb{R}^{\geq 0}$, so $\circ$ is indeed commutative.

   (b) Is $\circ$ associative?

   No, it isn't. Here is an example:

   $$\begin{aligned}(1 \circ 2) \circ 3 &= ||1 - 2| - 3| = |1 - 3| = 2 \\ 1 \circ (2 \circ 3) &= |1 - |2 - 3|| = |1 - 1| = 0\end{aligned}$$

   which are not equal.

   (c) Does $\circ$ have an identity?

   Yes, 0 works as an identity. If $x \in \mathbb{R}^{\geq 0}$, then

   $$\begin{aligned}x \circ 0 &= |x - 0| = |x| = x \\ 0 \circ x &= |0 - x| = |-x| = x\end{aligned}$$

   (d) Does every element in $\mathbb{R}^{\geq 0}$ have an inverse with respect to $\circ$?

   Yes. Every element is its own inverse. Let $x \in \mathbb{R}^{\geq 0}$. Then

   $$x \circ x = |x - x| = 0$$

3. (15 pts) Answer the following questions and carefully justify your answers.
   (a) True or false? An operation which has identity must be commutative.

   False. For example, we showed in class that composition of $\mathbb{R} \to \mathbb{R}$ functions is an operation with an identity (the identity function) but is not commutative.

(b) True or false? If an operation has an identity, then this identity is unique.

True. Suppose $\circ$ is an operation on S and $e, f \in S$ act as identities. Then $e \circ f = e$ because f is an identity and $e \circ f = f$ because e is an identity. Hence $e = f$.

(c) Let $\circ$ be an operation with an identity on the set S. Is it possible that no element of S has an inverse?

False. The identity is always its own inverse.

4. (15 pts) **Extra credit problem.** This is a hard problem. Attempt it only when you are done with everything else.

Let $\circ$ be an operation with identity on S. We say that an element $x \in S$ has *left inverse* y if $yx = e$. A *right inverse* of x is an element z such that $xz = e$. (Notice this is different from an inverse because xy and zx are not required to be e.)

Now, let $\circ$ be an associative operation with identity on S. Show that if every element of S has a left inverse, then every element of S also has a right inverse.

Let x be any element of S. We know x has some left inverse $y \in S$. So $yx = e$. We will show that y is also a right inverse of x.

Since every element of S has a left inverse, y also has a left inverse $z \in S$. So $zy = e$. Now

$$(zy)x = ex = x$$
$$z(yx) = ze = z$$

By associativity, these two are equal. Hence $z = x$ and $e = zy = xy$. We have just shown that y is also a right inverse of x.