Math 524 Homework 2

Due at 5:30 PM on 9/22

1. (a) Is $x \circ y = \max(x, y)$ on $\mathbb{Z}^+$ an operation? Is it commutative? Is it associative? Does it have an identity? Does every element have an inverse?
   (b) Is $x \circ y = \min(x, y)$ on $\mathbb{Z}^+$ an operation? Is it commutative? Is it associative? Does it have a identity? Does every element have an inverse?
   (c) Give an example of an associative, noncommutative operation with identity on the set S such that every element of S has an inverse. Make sure you justify your example.
2. Let $f : S \to T$ be a function. Remember that we say f has an *inverse* if there exists a function $f^{-1} : T \to S$ such that $f^{-1}(f(x)) = x$ for all $x \in S$ and $f(f^{-1}(y)) = y$ for all $y \in T$. A function that has an inverse is called *invertible*. In this exercise, you will prove that a function is invertible if and only if it is one-to-one and onto.
   (a) Show that if f is invertible then f is one-to-one. (Hint: use that $f^1 \circ f = \mathrm{Id}_S$.)
   (b) Show that if f is invertible then f is onto. (Hint: use f^{-1} to find a preimage of $y \in T$ for all y.)
   (c) Suppose f is one-to-one and onto. Construct a function $g : T \to S$ by sending each $y \in T$ to its preimage under f. Explain why you can do this. (Hint: how many preimages does an element of T have under f?) Now show that g is an inverse of f.
3. Let $\circ$ be an associative operation on S with identity e. Prove that if an element x has an inverse, then this inverse is unique (i.e. an element cannot have two different inverses).
4. **Extra credit problem.** Find an example of an operation with identity such that some element has more than one inverse. Note that by the previous exercise, such an operation must be nonassociative.