MATH 524 EXAM 1 SOLUTIONS

Oct 6, 2008

1. (10 pts) Let V be a vector space over the field F, $v \in V$ and $\alpha \in F$. Prove that if $\alpha v = 0$, then either $\alpha = 0$ or $v = 0$.

   If $\alpha = 0$, there is nothing to prove. So suppose $\alpha \neq 0$. Then
   $$v = (\alpha^{-1}\alpha)v = \alpha^{-1}(\alpha v) = \alpha^{-1}0 = 0.$$

2. (10 pts) Let V be a vector space over the field F. Prove that the intersection of any nonempty collection of subspaces of V is a subspace of V. (Keep in mind that a collection could be infinite.)

   Let $\{U_a\}$ be collection of subspaces, where a is an element of some indexing set A. Let $U = \cup_{a\in A} U_a$. We will prove U is a subspace of V. First, since U is a subset of any of the subspaces U_a, it is also a subset of V. Each U_a is a subspace and hence contains 0. Therefore $0 \in U$. Let $u, v \in U$. Then $u, v \in U_a$ for all $a \in A$. Since each U_a is a subspace, it is closed under addition and $u + v \in U_a$ for all $a \in A$. Hence $u + v \in U$. Finally, let $v \in U$ and $\alpha \in F$. Then $v \in U_a$ for all $a \in A$. Since each U_a is a subspace, it is closed under scalar multiplication and $\alpha v \in U_a$ for all $a \in A$. Hence $\alpha v \in U$.

3. (10 pts) Let V be a vector space over the field F. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that
   $$U_1 + W = U_2 + W$$
   then $U_1 = U_2$.

   Let $V = \mathbb{R}^2$ over $\mathbb{R}$, U_1 the x-axis, U_2 the y-axis and $W = V$. Then $U_1 + W = U_2 + W = V$. Here is why. That $U_1 + W, U_2 + W \subseteq V$ is clear since $U_1, U_2, W \subseteq V$. Let $v \in V$ any vector. Then $v \in W = V$. So $v = 0 + v \in U_1 + W$. Also, $v = 0 + v \in U_2 + W$. Hence $V \subseteq U_1 + W, U_2 + W$.

   But $U_1 \neq U_2$. So the statement is false in general.

4. (15 pts) Let V be a vector space over the field F and $U_1, U_2, \ldots, U_n$ subspaces of V. Prove that $U_1 + U_2 + \cdots + U_n$ is a direct sum if and only if the only way to write
   $$0 = u_1 + u_2 + \cdots + u_n$$
   with $u_i \in U_i$ is if $u_1 = u_2 = \cdots = u_n = 0$.

   If the sum is direct, any vector in it can be written as a sum $u_1 + \cdots + u_n$ in only one way. In particular, 0 can be written as such a sum in only one way, which is $u_1 = u_2 = \cdots = u_n = 0$.

   Conversely, suppose the only way to write 0 as $u_1 + \cdots + u_n$ is $u_1 = \cdots = u_n = 0$. Let $v \in U_1 + \cdots + U_n$. Suppose
   $$v = u_1 + u_2 + \cdots + u_n = v_1 + v_2 + \cdots + v_n$$
   with $u_i, v_i \in U_i$. Then
   $$\begin{aligned} 0 = v - v &= (u_1 + \cdots + u_n) - (v_1 + \cdots + v_n) \\ &= (\underbrace{u_1 - v_1}_{\in U_1}) + (\underbrace{u_2 - v_2}_{\in U_2}) + \cdots + (\underbrace{u_n - v_n}_{\in U_n}) \end{aligned}$$
   But 0 can be written as such a sum in only one way with $u_i - v_i = 0$. Hence $u_i = v_i$ and the two sum for v above must be one and the same.

5. (15 pts) Give an example of a vector space V (over some field which you clearly state) and subspaces U and W such that $V = U \oplus W$. Prove that U and W are indeed subspaces, $V = U + W$, and the sum is direct.

Let $V = \mathbb{R}^2$ over $\mathbb{R}$, U the x-axis, W the y-axis. Obviously, both U is a subset V and contains 0. Let $u_1, u_2 \in U$. That is there must exist $x_1, x_2 \in \mathbb{R}$ such that $u_1 = (x_1, 0)$ and $u_2 = (x_2, 0)$. Hence

$$u_1 + u_2 = (x_1, 0) + (x_2, 0) = (x_1 + x_2, 0) \in U.$$

If $\alpha \in \mathbb{R}$, then

$$\alpha u_1 = \alpha(x_1, 0) = (\alpha x_1, 0) \in U.$$

So U is a subspace of V. An analogous argument shows W is a subspace too.

Since $U, W \subseteq V$, $U + W \subseteq V$. Now, let $v \in V$. So there exist $x, y \in \mathbb{R}$ such that $v = (x, y)$. Then

$$v = (x, y) = (x, 0) + (0, y) \in U + W.$$

Hence $V \subseteq U + W$ and we can conclude $V = U + W$. Incidentally, it is clear that $(x, y) = (x, 0) + (0, y)$ is the only way to write the vector (x, y) as a sum of vectors from U and W, so $V = U \oplus W$.

6. (10 pts) **Extra credit problem.** Let $S = \{x_1, x_2, \ldots, x_n\} \subset \mathbb{R}$. Let

$$T = \{p \in \mathbb{R}[x] \mid p(x_i) = 0 \text{ for all } 1 \leq i \leq n\}$$

be the set of those polynomials with real coefficients which are 0 at every point in S. Is T a subspace of $\mathbb{R}[x]$? If so prove it, if not, explain why not.

Indeed, T is a subspace. Note that the zero polynomial is in T, since its value is 0 everywhere, including all numbers in S. Let $p, q \in T$. Then $p(x_i) = q(x_i) = 0$ for all $1 \leq i \leq n$. Hence $(p + q)(x_i) = p(x_i) + q(x_i) = 0$ for all i. This shows $p + q \in T$. Finally, let $p \in T$ and $\alpha \in \mathbb{R}$. We know $p(x_i) = 0$ for all i. Hence $(\alpha p)(x_i) = \alpha p(x_i) = \alpha 0 = 0$, which shows $\alpha p \in T$. Therefore T is a subspace.