Math 524 Exam 2 Solutions

Nov 3, 2008

1. (10 pts) Let V be a vector space. Prove that if $(v_1, \ldots, v_n)$ spans V, then so does the list
$$(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$$
obtained by subtracting from each vector (except the last one) the following vector.

   Let $U = \operatorname{span}(v_1 - v_2, \ldots, v_{n-1} - v_n, v_n)$. Obviously, $U \subseteq V$. Note that for $1 \le i \le n$,
$$v_i = (v_i - v_{i+1}) + (v_{i+1} - v_{i+2}) + \cdots + (v_{n-1} - v_n) + v_n.$$
So v_i is a linear combination of $v_1 - v_2, \ldots, v_{n-1} - v_n, v_n$ and hence is in U. Since U is a subspace, it is closed under linear combinations. In particular, all linear combinations of $v_1, \ldots, v_n$ are in U. Hence $V = \operatorname{span}(v_1, \ldots, v_n) \subseteq U$.

2. (10 pts) Let V be a vector space. Prove that V is infinite dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $(v_1, \ldots, v_n)$ is linearly independent for every positive integer n.

   First, suppose $\dim(V) = \infty$. Build a sequence of vectors as follows. For v_1, choose any nonzero vector in V. You can do this, since $V \neq \{0\}$ (otherwise it would be 0-dimensional). Since $\dim(V) = \infty$, $V \neq \operatorname{span}(v_1)$. So pick $v_2 \notin \operatorname{span}(v_1)$. Again, $V \neq \operatorname{span}(v_1, v_2)$. So pick $v_3 \notin \operatorname{span}(v_1, v_2)$. And so on. This gives an infinite list $v_1, v_2, \ldots$ in which none of the vectors is in the span of the preceding ones. Let $n \in \mathbb{Z}^+$. Then $(v_1, \ldots, v_n)$ must be linearly independent, otherwise there would be a $1 \le j \le n$ such that $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$.

   Now, suppose that we have a sequence $v_1, v_2, \ldots$ with the property above. Suppose that V is not infinite dimensional. Then V is spanned by some finite list of vectors $(u_1, \ldots, u_k)$. We know that any spanning list must contain at least as many vectors as any linearly independent list, so $(v_1, \ldots, v_{k+1})$ cannot be linearly independent. This is a contradiction. Hence V must be infinite dimensional.

3. (10 pts) Let V be a finite dimensional vector space and B_1 and B_2 bases of V. Prove that $|B_1| = |B_2|$.

   See the proof of Theorem 2.14 in the textbook.

4. (10 pts) Let U be the subspace pf $\mathbb{R}^5$ defined by
$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$
Find a basis of U. (And prove that it is a basis.)

   Notice that any vector in $u \in U$ is of the form $(3x_2, x_2, 7x_4, x_4, x_5)$ where $x_2, x_4, x_5 \in \mathbb{R}$. Therefore
$$u = x_2(3, 1, 0, 0, 0) + x_4(0, 0, 7, 1, 0) + x_5(0, 0, 0, 0, 1)$$
and
$$U = \operatorname{span}\left((3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)\right).$$
Observe that
$$\begin{aligned}(0, 0, 0, 0, 0) &= \alpha_1(3, 1, 0, 0, 0) + \alpha_2(0, 0, 7, 1, 0) + \alpha_3(0, 0, 0, 0, 1)\\ &= (3\alpha_1, \alpha_1, 7\alpha_2, \alpha_2, \alpha_3)\end{aligned}$$
implies $\alpha_1 = \alpha_2 = \alpha_3 = 0$. So these vectors are linearly independent. Therefore they form a basis of U.

5. (20 pts)
   (a) Define what a finite dimensional vector space is.

   A vector space V is finite dimensional if there are vectors $v_1, \ldots, v_n \in V$ such that $V = \text{span}(v_1, \ldots, v_n)$.

   (b) Give an example of an infinite dimensional vector space V (over some field F which you should be sure to specify). Prove that V is infinite dimensional.

   Consider $\mathbb{R}^\infty$, the vector space of infinite real sequences over the field $\mathbb{R}$. Let
$$v_n = (\underbrace{0, \ldots, 0}_{n-1}, 1, 0, \ldots).$$
   Notice that for any n, $v_n \notin \text{span}(v_1, \ldots, v_{n-1})$, since the n-th coordinates of $v_1, \ldots, v_{n-1}$ are all 0. Therefore $(v_1, \ldots, v_n)$ is linearly independent for any $n \in \mathbb{Z}^+$. By problem 2, $\dim_{\mathbb{R}}(\mathbb{R}^\infty) = \infty$.

   (c) Let V be a finite dimensional vector space. Define what a basis of V is.

   A basis of V is a list of vectors $(v_1, \ldots, v_n)$ in V which is linearly independent and spans V.

   (d) Give an example of a finite dimensional vector space V (again, do not forget to specify the base field F) and a basis B of V. Prove that B is indeed a basis of V.

   Let $V = \mathbb{Q}^2$ and $F = \mathbb{Q}$. Then $B = ((1, 0), (0, 1))$ is a basis of V. First, any vector in V is of the form (x, y) where $x, y \in \mathbb{Q}$. Since $(x, y) = x(1, 0) + y(0, 1)$, it is in the span of B. If
$$(0, 0) = \alpha(1, 0) + \beta(0, 1) = (\alpha, \beta)$$
   then $\alpha = \beta = 0$, so B is linearly independent. Therefore it is a basis of V.

6. (15 pts) **Extra credit problem.** Let V be a vector space over some field F and $U = (u_1, u_2, \ldots)$ some infinite set of vectors in V. Define a linear combination of U as a sum
$$\sum_{i=1}^{\infty} \alpha_i u_i$$
such that $\alpha_i \in F$ and only finitely many of the α_i are nonzero. For example, let $F = \mathbb{R}$ and $V = \mathbb{R}[x]$, the vector space of polynomials with real coefficients, and $U = (1, x, x^2, \ldots)$. Then
$$2x^3 - 5x + 3 = 3(1) + 5x + 0x^2 + 2x^3 + 0x^4 + 0x^5 + \cdots$$
is a linear combination of U, but
$$1 + x + x^2 + x^3 + x^4 + \cdots$$
is not.

   (a) Let V be a vector space over some field F and $U = (u_1, u_2, \ldots)$ some infinite set of vectors in V. As usual, define span(U) to be the set of all linear combinations of U. Prove that span(U) is a subspace of V.

   First, note that
$$0 = 0u_1 + 0 + 0u_2 + 0u_3 + \cdots$$
   is indeed a linear combination of $(u_1, u_2, \ldots)$ as only finitely many (0 is a finite number!) of the coefficients are nonzero. So $0 \in \text{span}(U)$.

Now let $v, w \in \text{span}(U)$. So

$$v = \sum_{i=1}^{\infty} \alpha_i u_i, \qquad w = \sum_{i=1}^{\infty} \beta_i u_i$$

where only finitely many of $\alpha_1, \alpha_2, \dots$ and $\beta_1, \beta_2, \dots$ are nonzero. Therefore there is a largest index m such that $\alpha_m \neq 0$ (if all $\alpha_i = 0$, set $m = 0$) and a largest n such that $\beta_n \neq 0$. So $\alpha_i = 0$ for $m < i$ and $\beta_i = 0$ for $n < i$. Let $k = \max(m, n)$. Then

$$v = \sum_{i=1}^{k} \alpha_i u_i, \qquad w = \sum_{i=1}^{k} \beta_i u_i$$

since $\alpha_i = \beta_i = 0$ for $k < i$. Hence

$$\begin{aligned} v + w &= \sum_{i=1}^{k} \alpha_i u_i + \sum_{i=1}^{k} \beta_i u_i = \sum_{i=1}^{k} (\alpha_i + \beta_i) u_i \\ &= (\alpha_1 + \beta_1) u_1 + \cdots + (\alpha_k + \beta_k) u_k + 0 u_{k+1} + 0 u_{k+2} + \cdots \end{aligned}$$

which is a linear combination of U, as all but the first k of the coefficients are 0. Finally, let $v \in \text{span}(U)$ and $\beta \in F$. So

$$v = \sum_{i=1}^{\infty} \alpha_i u_i$$

where only finitely many of $\alpha_1, \alpha_2, \dots$ are nonzero. Therefore there is a largest index m such that $\alpha_m \neq 0$. So $\alpha_i = 0$ for $m < i$ and

$$v = \sum_{i=1}^{k} \alpha_i u_i$$

Hence

$$\begin{aligned} \beta v &= \beta \sum_{i=1}^{m} \alpha_i u_i = \sum_{i=1}^{m} (\beta \alpha_i) u_i \\ &= (\beta \alpha_1) u_1 + \cdots + (\beta \alpha_m) u_m + 0 u_{m+1} + 0 u_{m+2} + \cdots \end{aligned}$$

which is a linear combination of U as all but the first m of the coefficients are 0.

(b) Let $F = \mathbb{R}$ and $V = \mathbb{R}[x]$, the vector space of polynomials with real coefficients, and $U = (1, x, x^2, \dots)$. Prove that $V = \text{span}(U)$.

Let $p \in V$. Then $p(x) = a_0 + a_1 x + \cdots a_n x^n$ for some $n \in \mathbb{Z}^+$ and $a_0, \dots, a_n \in \mathbb{R}$. So

$$p(x) = a_0 + a_1 x + \cdots a_n x^n + 0 x^{n+1} + 0 x^{n+2} + \cdots,$$

which is indeed a linear combination of $(1, x, x^2, \dots)$ as all but the first $n+1$ coefficients are 0.

(c) As usual, define U to be linearly independent, if the only linear combination of vectors equal to 0 is the one in which all coefficients are 0. Let $F = \mathbb{R}$ and $V = \mathbb{R}[x]$, the vector space of polynomials with real coefficients, and $U = (1, x, x^2, \dots)$. Prove that U is linearly independent.

Let $\alpha_0, \alpha_1, \dots \in F$ be such that all finitely many of them are 0 and

$$0 = \sum_{n=0}^{\infty} \alpha_n x^n$$

Again, there is some $m \in \mathbb{N}$ such that $\alpha_i = 0$ for all $m < i$. Therefore

$$0 = \sum_{n=0}^{m} \alpha_n x^n.$$

The RHS is actually a polynomial, so it is 0 if all of its coefficients are 0. Hence $\alpha_0 = \alpha_1 = \cdots = \alpha_m = 0$. Since all of the coefficients must be 0, U is linearly independent.