Math 524 Exam 3 Solutions

Nov 26, 2008

1. (10 pts) Let V and W be vector spaces over some field F and $T \in \mathcal{L}(V, W)$. Prove that if T is injective and $(v_1, \ldots, v_n)$ is linearly independent in V, then $(T(v_1), \ldots, T(v_n))$ is linearly independent in W.

   Let $\alpha_1, \ldots, \alpha_n \in F$ be such that
   $$\alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) = 0.$$
   Since T is linear
   $$0 = \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) = T(\alpha_1 v_1 + \cdots + \alpha_n v_n).$$
   But T is also injective, so the only vector in its null space is 0. Hence
   $$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0.$$
   This shows $\alpha_1 = \cdots = \alpha_n = 0$ by the linear independence of $(v_1, \ldots, v_n)$.

2. (10 pts) Suppose V and W are both finite dimensional vector spaces over the same field F. Prove that there exists a surjective linear map $T : V \to W$ if and only if $\dim(W) \leq \dim(V)$.

   Suppose $T \in (V, W)$ is surjective. Then
   $$\dim(V) = \dim \operatorname{null}(T) + \dim R(T) = \dim \operatorname{null}(T) + \dim(W).$$
   Since $\dim \operatorname{null}(T) \geq 0$, it must be that $\dim(W) \leq \dim(V)$.

   Conversely, suppose $\dim(W) \leq \dim(V)$. Choose a basis $v_1, \ldots, v_m$ for V and $w_1, \ldots, w_n$ for W. Construct a linear map $T : V \to W$ by letting $T(v_i) = w_i$ for $1 \leq i \leq n$ and $T(v_i) = 0$ for $n < i$. Extend T to other vectors in V by linearity, i.e. $T(\sum \alpha_i v_i) = \sum \alpha_i T(v_i)$. (See the bottom of p. 39 and top of p. 40 in your text on this.)

   We need to show T is surjective. Let $w \in W$. Then there exist $\beta_1, \ldots, \beta_n$ such that $w = \beta_1 w_1 + \cdots + \beta_n w_n$. Let $v = \beta_1 v_1 + \cdots + \beta_n v_n$ and notice
   $$T(v) = T(\beta_1 v_1 + \cdots + \beta_n v_n) = \beta_1 T(v_1) + \cdots + \beta_n T(v_n) = \beta_1 w_1 + \cdots + \beta_n w_n = w.$$

3. (10 pts) Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose A, B, and C are matrices whose sizes are such that $A(B+C)$ makes sense. Prove that $AB + AC$ makes sense and $A(B + C) = AB + AC$.

   For $A(B+C)$ to make sense, the number of rows of $B+C$ must be the same as the number of columns of A. Call this number m. For $B + C$ to make sense, B and C must have the same size. In particular, they must have the same number of rows, which is also the number of rows in $B+C$, that is m. So B and C have m rows and A has m columns, hence AB and AC make sense.

   Let k be the number of rows of A and n be the number of columns of B and C. Then
   $$\begin{aligned}(A(B+C))_{ij} &= \sum_{l=1}^{m} A_{im}(B+C)_{mj} = \sum_{l=1}^{m} A_{im}(B_{mj} + C_{mj}) \\ &= \sum_{l=1}^{m} A_{im}B_{mj} + A_{im}C_{mj} = \sum_{l=1}^{m} A_{im}B_{mj} + \sum_{l=1}^{m} A_{im}C_{mj} \\ &= (AB)_{ij} + (AC)_{ij}\end{aligned}$$
   This is true for all $1 \leq i \leq k$ and $1 \leq j \leq n$, so $A(B+C) = AB + AC$.

4. (10 pts) Let V and W be vector spaces over some field F. Let $T : V \to W$ be a linear map. Prove that if T has an inverse $S : W \to V$, then S is linear too.

Let $w_1, w_2 \in W$. Then

$$\begin{aligned} TS(w_1 + w_2) &= 1_W(w_1 + w_2) = w_1 + w_2 = 1_W(w_1) + 1_W(w_2) \\ &= TS(w_1) + TS(w_2) = T(S(w_1) + S(w_2)) \end{aligned}$$

Since T has an inverse, T is injective, and hence $S(w_1 + w_2) = S(w_1) + S(w_2)$.

Now, let $w \in W$ and $\alpha \in F$. Then

$$TS(\alpha w) = 1_W(\alpha w) = \alpha w = \alpha\, 1_W(w) = \alpha TS(w) = T(\alpha S(w))$$

By the injectivity of T, $S(\alpha w) = \alpha S(w)$.

5. (10 pts each) Let $V = \mathbb{R}^3$ and $W = \mathbb{R}^4$ over $\mathbb{R}$. Let

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - z \\ x - y \\ x - 3y + z \\ z - 2y \end{pmatrix}.$$

(a) Find the null space and range of T.

$$0 = T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - z \\ x - y \\ x - 3y + z \\ z - 2y \end{pmatrix}$$

implies $z = 2x$, $y = x$ from the first two equations. Notice that if these are satisfied

$$x - 3y + z = x - 3x + 2x = 0 \text{ and } z - 2y = 2x - 2x = 0.$$

So

$$\text{null}(T) = \left\{ \begin{pmatrix} x \\ x \\ 2x \end{pmatrix} \mid x \in \mathbb{R} \right\} = \left\{ x\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \mid x \in \mathbb{R} \right\} = \text{span}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

As for the range

$$\begin{aligned} T(V) &= T\left(\text{span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)\right) \\ &= \text{span}\left(T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) \\ &= \text{span}\left(\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -3 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}\right) \end{aligned}$$

By the dimension theorem, we expect $\dim R(T) = \dim(V) - \dim \operatorname{null}(T) = 3 - 1 = 2$. Indeed, notice that

$$\begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ -3 \\ -2 \end{pmatrix}\right)$$

So

$$R(T) = \operatorname{span}\left(\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -3 \\ -2 \end{pmatrix}\right) = \left\{ x\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ -1 \\ -3 \\ -2 \end{pmatrix} \mid x, y \in \mathbb{R} \right\}.$$

(b) Let

$$B_1 = \left(\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}\right).$$

and

$$B_2 = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}\right).$$

Notice that B_1 is a basis of V and B_2 is a basis of W. Find the matrix of T with respect to these bases.

We need to write the three vectors

$$T\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -6 \\ -4 \end{pmatrix}, \qquad T\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \\ -2 \\ 0 \end{pmatrix}, \qquad T\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \\ -1 \end{pmatrix}$$

as linear combinations of B_2. So

$$\begin{pmatrix} 0 \\ -2 \\ -6 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

Since B_2 is a basis, the 4 by 4 matrix above must have an inverse. It will help if we find it first:

$$\left(\begin{array}{cccc|cccc} 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & \frac{1}{2} & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \end{array}\right)$$

Now

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \\ -6 \\ -4 \end{pmatrix} = \begin{pmatrix} -\frac{9}{2} \\ -5 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} -4 \\ -2 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{5}{2} \\ -1 \\ \frac{5}{2} \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ -1 \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

Hence

$$M(T) = \begin{pmatrix} -\frac{9}{2} & -\frac{5}{2} & -\frac{3}{2} \\ -5 & -1 & -1 \\ \frac{1}{2} & \frac{5}{2} & \frac{1}{2} \\ -1 & -1 & 0 \end{pmatrix}.$$

6. (20 pts) **Extra credit problem.** Let U, V, and W be vector spaces over some field F such that U and V are finite dimensional. Let $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. In this problem, you will prove that

$$\dim \operatorname{null}(ST) \leq \dim \operatorname{null}(S) + \dim \operatorname{null}(T).$$

(You may find that the proof is very similar to the proof of the Dimension Theorem. If you don't know how to prove one of the statements below, feel free to move on the next one and assume that all the preceding statements have been proven.)

(a) Prove that $\operatorname{null}(T) \subseteq \operatorname{null}(ST)$.

Let $u \in \operatorname{null}(T)$. Then $ST(u) = S(0) = 0$. Hence $u \in \operatorname{null}(ST)$. This is true for all $u \in \operatorname{null}(T)$, so $\operatorname{null}(T) \subseteq \operatorname{null}(ST)$.

(b) Show that there exists a basis $(u_1, \ldots, u_n)$ of null(ST) and some $0 \leq m \leq n$ such that $(u_1, \ldots u_m)$ is a basis of null(T).

First, note that null(T) and null(ST) are subspaces of U, which is a finite dimensional vector space. Therefore null(T) and null(ST) are also finite dimensional. So we can pick a basis $(u_1, \ldots u_m)$ of null(T). This is a linearly independent list in null(T) and hence null(ST) and can therefore be extended to a basis $(u_1, \ldots, u_n)$ of null(ST).

(c) Prove that $(T(u_{m+1}), \ldots, T(u_n))$ is linearly independent in V.

Suppose $\alpha_{m+1}, \ldots, \alpha_n \in F$ are such that

$$\alpha_{m+1}T(u_{m+1}) + \cdots + \alpha_n T(u_n) = 0.$$

Then

$$T(\alpha_{m+1}u_{m+1} + \cdots + \alpha_n u_n) = 0.$$

Therefore

$$\alpha_{m+1}u_{m+1} + \cdots + \alpha_n u_n \in \text{null}(T) = \text{span}(u_1, \ldots u_m).$$

Hence there exist $\beta_1, \ldots, \beta_m \in F$ such that

$$\alpha_{m+1}u_{m+1} + \cdots + \alpha_n u_n = \beta_1 u_1 + \cdots + \beta_m u_m.$$

By the linear independence of $(u_1, \ldots, u_n)$, all the α_i and β_j must be 0. Therefore $(T(u_{m+1}), \ldots, T(u_n))$ is linearly independent.

(d) Prove that $(T(u_{m+1}), \ldots, T(u_n))$ is linearly independent in null(S).

Linear independence is not really an issue here. We already $(T(u_{m+1}), \ldots, T(u_n))$ is linearly independent in V, therefore it is linearly independent in any subspace of V in which it is contained. The question is whether $T(u_{m+1}), \ldots, T(u_n) \in$ null(S). Since $u_i \in$ null(ST), therefore $S(T(u_i)) = 0$, so $u_i \in$ null(S) indeed. Therefore $(T(u_{m+1}), \ldots, T(u_n))$ is a linearly independent list in null(S).

(e) Show that $n - m \leq \dim \text{null}(S)$. Conclude that

$$\dim \text{null}(ST) \leq \dim \text{null}(S) + \dim \text{null}(T).$$

Since $(T(u_{m+1}), \ldots, T(u_n))$ is a linearly independent list in null(S), its length must be no more than $\dim \text{null}(S)$. But its length is exactly $n - m$. Hence $n - m \leq \dim \text{null}(S)$. Hence

$$n \leq \dim \text{null}(S) + m \implies \dim \text{null}(ST) \leq \dim \text{null}(S) + \dim \text{null}(T).$$