Math 524 Final Exam Solutions

Dec 19, 2008

1. (10 pts) Let V be a vector space. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that
$$V = U_1 + W \quad \text{and} \quad V = U_2 + W$$
then $U_1 = U_2$.

   This problem was on your first exam. See its solution there.

2. (10 pts) Let F be any field. Recall that
$$F^\infty = \{(x_1, x_2, \ldots) \mid x_i \in F \text{ for } i = 1, 2, \ldots\}$$
is the vector space of all sequences with elements in F over the base field F. Prove that F^∞ is infinite dimensional.

   The proof is by contradiction. Suppose F^∞ is finite dimensional. Let $n = \dim(F^\infty)$. For $i = 1, 2, \ldots, n+1$, let v_i be the sequence whose i-th entry is 1 and all of its other entries are 0. E.g.
$$v_3 = (0, 0, 1, 0, 0, \ldots).$$
Then the only solution to
$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_{n+1} v_{n+1} = (0, 0, 0, \ldots)$$
is $\alpha_1 = \ldots = \alpha_{n+1} = 0$, since the only way to get 0 in the i-th entry is to multiply v_i by 0. Hence $(v_1, \ldots, v_{n+1})$ is linearly independent. But the length of a linearly independent list cannot exceed the dimension in a finite dimensional vector space. We have reached a contradiction.

3. (10 pts) Let V be a vector space. Prove that if there exists a linear map from V to some other vector space, whose null space and range are both finite dimensional, then V is finite dimensional. (Hint: It is tempting to use the Dimension Theorem, but that doesn't quite work.)

   This proof is also by contradiction. Suppose V is infinite dimensional and T is a linear map from V to some vector space W, such that $\text{null}(T)$ and $R(T)$ are both finite dimensional. Let $n = \dim \text{null}(T)$ and $k = \dim R(T)$. Choose a basis $(v_1, \ldots, v_n)$ for $\text{null}(T)$. Now choose $v_{n+1} \in V$ so that $v_{n+1} \notin \text{span}(v_1, \ldots, v_n)$. Choose $v_{n+2} \in V$ so that $v_{n+2} \notin \text{span}(v_1, \ldots, v_{n+1})$. Keep doing this until you have chosen v_{n+k+1}. The reason you can always find such vectors is that V is infinite dimensional, therefore no finite list can span it, therefore there is always a vector outside $\text{span}(v_1, \ldots, v_j)$. Obviously, $(v_1, \ldots, v_{n+k+1})$ is linearly independent, since none of the vectors in this list is a linear combination of the preceding ones. We will prove that $(T(v_{n+1}), \ldots, T(V_{n+k+1}))$ is linearly independent in W. Suppose
$$\sum_{i=n+1}^{n+k+1} \alpha_i T(v_i) = 0.$$
Then
$$T\left(\sum_{i=n+1}^{n+k+1} \alpha_i v_i\right) = 0,$$

so $\sum_{i=n+1}^{n+k+1} \alpha_i v_i \in \text{null}(T)$. Therefore there exist $\beta_1, \dots, \beta_n \in F$ such that

$$\sum_{i=n+1}^{n+k+1} \alpha_i v_i = \sum_{i=1}^{n} \beta_i v_i.$$

This is only possible if all the α_i and β_i are 0 because of the linear independence of $(v_1, \dots, v_{n+k+1})$. Hence $(T(v_{n+1}), \dots, T(V_{n+k+1}))$ is indeed linearly independent. But

$$T(v_{n+1}), \dots, T(V_{n+k+1}) \in T(V) = R(T)$$

which is only k-dimensional, hence it cannot contain a linearly independent list of $k+1$ vectors. That is a contradiction.

4. (10 pts) Let V be a nonzero finite dimensional vector space and $S, T \in \mathcal{L}(V, V)$. Prove that ST and TS have the same eigenvalues.

Suppose λ is an eigenvalue of ST and v is a corresponding eigenvector. Then

$$TS(T(v)) = (TS)T(v) = T(ST)(v) = T(\lambda v) = \lambda T(v).$$

This shows that if $T(v) \neq 0$, then $T(v)$ is an eigenvector of TS with eigenvalue λ. In fact, if $T(v) = 0$, then $ST(v) = 0$, hence $\lambda = 0$. So if $\lambda \neq 0$, then $T(v)$ is an eigenvector of TS with eigenvalue λ. This shows that any nonzero eigenvalue of ST is also an eigenvalue of TS. The argument is symmetric, so any nonzero eigenvalue of TS must also be an eigenvalue of ST. Hence the nonzero eigenvalues of ST and TS are the same.

Now if 0 is an eigenvalue of ST, then there is a nonzero vector $v \in V$ such that $ST(v) = 0$. If $T(v) \neq 0$, then it is also an eigenvector of TS with eigenvalue 0 by the argument in the previous paragraph. On the other hand, if $T(v) = 0$, then T is not one-to-one. Since V is finite dimensional, T cannot be onto either since V (Theorem 3.21). Therefore TS is not onto (why?), and TS is not one-to-one (again by Theorem 3.21). Hence $\text{null}(TS) \neq \{0\}$, and any nonzero vector in $\text{null}(TS)$ is an eigenvector of TS with eigenvalue 0. This argument is symmetric too, so 0 is an eigenvalue of ST if and only if it is an eigenvalue of TS.

Remark: The statement is not true if V is allowed to be infinite dimensional. Can you find a counterexample? There is one in your textbook and we looked at it in class too.

5. (10 pts) Let V be a vector space over the field F. Prove that the list $(v_1, \dots, v_n)$ is a basis of V if and only if every $v \in V$ can be written as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

for some unique $\alpha_1, \dots, \alpha_n \in F$.

This is Proposition 2.8 in your text and is proven there.

6. (10 pts) Prove that matrix multiplication is associative. That is suppose A, B, and C are matrices with entries in some field F such that $(AB)C$ makes sense. Prove that $A(BC)$ makes sense and that $(AB)C = A(BC)$.

Let $A \in M_{m\times n}(F)$. Then for AB to make sense, $B \in M_{n\times r}(F)$. This gives $AB \in M_{m\times r}(F)$. For $(AB)C$ to make sense $C \in M_{r\times s}(F)$, which gives $(AB)C \in M_{m\times s}(F)$. Since $B \in M_{n\times r}(F)$ and $C \in M_{r\times s}(F)$, BC makes sense and is in $M_{n\times s}(F)$. Finally, $A(BC)$

makes sense and is in $M_{m\times s}(F)$ just like $(AB)C$. Now that we know $(AB)C)$ and $A(BC)$ are the same size, we just need to compare the their entries.

$$((AB)C)_{ij} = \sum_{k=1}^{r}(AB)_{ik}C_{kj} = \sum_{k=1}^{r}\left(\sum_{l=1}^{n} A_{il}B_{lk}\right)C_{kj} = \sum_{k=1}^{r}\sum_{l=1}^{n} A_{il}B_{lk}C_{kj}$$

$$(A(BC))_{ij} = \sum_{l=1}^{n} A_{il}(BC)_{lj} = \sum_{l=1}^{n} A_{il}\left(\sum_{k=1}^{r} B_{lk}C_{kj}\right) = \sum_{l=1}^{n}\sum_{k=1}^{r} A_{il}B_{lk}C_{kj}$$

Since the order of the summations is interchangeable (addition is commutative!), these are indeed the same.

7. (20 pts) Let $V = \mathbb{R}[x]$ be the vector space of polynomials over $\mathbb{R}$. Let $T = \frac{d}{dx} : V \to V$ be the usual differentiation map.
   (a) (4 pts) Prove that T is linear.

   Let $p, q \in V$ and $c \in \mathbb{R}$. By the usual properties of differentiation you learned in calculus,

$$T(cp) = \frac{d}{dx}(cp) = c\frac{dp}{dx} = cT(p)$$

$$T(p+q) = \frac{d}{dx}(p+q) = \frac{dp}{dx} + \frac{dq}{dx} = T(p) + T(q)$$

   Hence T is linear.

   (b) (5 pts) Is T one-to-one? Find null(T).

   T is definitely not one-to-one. For example, $T(2) = 0$. In fact, you learned in calculus that the derivative of a function is 0 if and only if that function is constant. Hence

$$\text{null}(T) = \{c \in \mathbb{R}\} = \mathbb{R}$$

   that is null(T) is the set of constant polynomials.

   (c) (5 pts) Is T onto? Find $R(T)$.

   T is onto. Any polynomial has an antiderivative. Let $p(x) = a_n x^n + \ldots + a_1 x + a_0 \in \mathbb{R}[x]$. Then

$$q(x) = \frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n}x^n + \ldots + \frac{a_1}{2}x^2 + a_0 x$$

   is also in $\mathbb{R}[x]$. Notice that $T(q) = p$. Hence $p \in R(T)$. You can do this for any polynomial p, so $\mathbb{R}[x] \subseteq R(T)$. Obviously $R(T) \subseteq \mathbb{R}[x]$ as well, so $R(T) = \mathbb{R}[x]$.

   (d) (2 pts) Does T have an inverse? If so, find it.

   Since T is not one-to-one, it cannot have an inverse.

   (e) (4 pts) Find all eigenvalues of T.

   Since null$(T) \neq \{0\}$, one of the eigenvalues of T is 0. For example, $T(2) = 0 \cdot 2$. In fact, any nonzero constant polynomial is an eigenvector of T with eigenvalue 0. We will show that 0 is the only eigenvalue. Suppose $p \in \mathbb{R}[x]$ is nonconstant. Since $\deg(T(p)) = \deg(p) - 1$, $T(p)$ cannot possibly be a scalar multiple of p. So p is not an eigenvector. Hence the only eigenvectors are the nonzero constant polynomials, and they correspond to eigenvalue 0.

8. (10 pts) **Extra credit problem.** Let $p \in \mathbb{C}[x]$ be of degree m. Prove that p has m distinct roots if and only if p and its derivative p' have no roots in common.

By the Fundamental Theorem of Algebra, p is a product of m linear factors

$$p(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_m)$$

where $\alpha_1, \ldots, \alpha_m$ are the roots of p.

First, suppose that $\alpha_1, \ldots, \alpha_m$ are distinct. We get p' by using the product rule:

$$p'(x) = (x - \alpha_2) \cdots (x - \alpha_m) + (x - \alpha_1)(x - \alpha_3) \cdots (x - \alpha_m) + \cdots + (x - \alpha_1) \cdots (x - \alpha_{m-1}).$$

Now

$$p'(\alpha_1) = (\alpha_1 - \alpha_2) \cdots (\alpha_1 - \alpha_m)$$

since all the other terms in the sum have a factor of $\alpha_1 - \alpha_1 = 0$. Since $\alpha_1 \neq \alpha_i$ for $2 \leq i \leq m$, $p'(\alpha_1) \neq 0$. A similar argument shows $p'(\alpha_j) \neq 0$ for all $j = 2, \ldots, m$. Hence none of the roots of p are roots of p'.

Conversely, suppose at least two of the roots are equal. Without loss of generality, $\alpha_1 = \alpha_2$. (You can always reorder the roots so that two equal ones are α_1 and α_2.) Then

$$p(x) = (x - \alpha_1)^2 \underbrace{(x - \alpha_3) \cdots (x - \alpha_m)}_{q(x)}.$$

So

$$\begin{aligned} p'(x) &= \frac{d}{dx}(x - \alpha_1)^2 q(x) \\ &= 2(x - \alpha_1)q(x) + (x - \alpha_1)^2 q'(x) \\ &= (x - \alpha_1)\left(2q(x) + (x - \alpha_1)q'(x)\right). \end{aligned}$$

This shows that α_1 is also a root of p'.

9. **Extra credit problem.** Let V be a nonzero vector space over some field F and $T \in \mathcal{L}(V, V)$. For a scalar $\lambda \in F$, define

$$U_\lambda = \{v \in V \mid T(v) = \lambda v\}.$$

(a) (6 pts) Prove that U_λ is a subspace of V and is invariant under T.

First, notice that $0 \in U_\lambda$ since $T(0) = 0 = \lambda 0$ for any λ. Now, let $v, w \in U_\lambda$ and $\alpha \in F$. Then

$$T(v + w) = T(v) + T(w) = \lambda v + \lambda w = \lambda(v + w),$$

so $v + w \in U_\lambda$. Similarly

$$T(\alpha v) = \alpha T(v) = \alpha(\lambda v) = \lambda(\alpha v),$$

so $\alpha v \in U_\lambda$. Hence U_λ is a subspace of V.

Now, if $v \in U_\lambda$, then $T(v) = \lambda v \in U_\lambda$, hence $T(U_\lambda) \subseteq U_\lambda$. Thus U_λ is indeed invariant under T.

(b) (4 pts) Prove that $U_\lambda \neq \{0\}$ if and only if λ is an eigenvalue of T.

If $U_\lambda \neq \{0\}$ then there exists a nonzero $v \in U_\lambda$. By definition of U_λ, v must satisfy $T(v) = \lambda v$, hence v is an eigenvector of T and λ is an eigenvalue.

Conversely, if λ is an eigenvalue of T, then there exists an eigenvector v such that $T(v) = \lambda v$. Since eigenvectors are nonzero, $0 \neq v \in U_\lambda$ shows $U_\lambda \neq \{0\}$ in this case.