Math 524 Exam 1 Solutions

Feb 16, 2011

1. (10 pts) Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of $\mathbb{R}^2$.

   One such is $U = \mathbb{Z}^2$. Notice that $U \subseteq \mathbb{R}^2$. Let $(x_1, x_2), (y_1, y_2) \in U$. Then $x_1, x_2, y_1, y_2 \in \mathbb{Z}$. Hence $x_1 + x_2, y_1 + y_2 \in \mathbb{Z}$ since $\mathbb{Z}$ is closed under addition. Thus

   $$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \in \mathbb{Z}^2.$$

   Also, if $(x_1, x_2) \in \mathbb{Z}^2$, then $x_1, x_2 \in \mathbb{Z}$, so $-x_1, -x_2 \in \mathbb{Z}$, and hence $-(x_1, x_2) = (-x_1, -x_2) \in \mathbb{Z}^2$. So $U = \mathbb{Z}^2$ is closed under addition and additive inverses.

   But U is not closed under scalar multiplication. For example, $1/2 \in \mathbb{R}$ and $(1, 1) \in U$, but $(1/2)(1, 1) = (1/2, 1/2) \notin U$.

2. (10 pts) Let V be a vector space over the field F. Prove that the intersection of any collection of subspaces of V is a subspace of V. (Keep in mind that a collection could be infinite.)

   Let $\{U_a\}$ be collection of subspaces, where a is an element of some indexing set A. Let $U = \cup_{a \in A} U_a$. We will prove U is a subspace of V. First, since U is a subset of any of the subspaces U_a, it is also a subset of V. Each U_a is a subspace and hence contains 0. Therefore $0 \in U$. Let $u, v \in U$. Then $u, v \in U_a$ for all $a \in A$. Since each U_a is a subspace, it is closed under addition and $u + v \in U_a$ for all $a \in A$. Hence $u + v \in U$. Finally, let $v \in U$ and $\alpha \in F$. Then $v \in U_a$ for all $a \in A$. Since each U_a is a subspace, it is closed under scalar multiplication and $\alpha v \in U_a$ for all $a \in A$. Hence $\alpha v \in U$.

3. (10 pts) Does the operation of addition on subspaces have an additive identity? Which subspaces have additive inverses? (Be sure to justify your answers.)

   There is in fact an additive identity, it is the 0 space. Let V be a vector space. If $U \subseteq V$ is a subspace, then we will show that $U + \{0\} = U$ and $\{0\} + U = U$. First, if $u \in U$, then $u = u + 0 \in U + \{0\}$. Hence $U \subseteq U + \{0\}$. On the other hand, any element of $U + \{0\}$ is of the form $u + 0$ where $u \in U$, and obviously $u + 0 = u \in U$. Hence $U + \{0\} \subseteq U$. Therefore $U + \{0\} = U$. An analogous argument shows $\{0\} + U = U$.

   The only subspace that has an additive inverse is the 0 subspace. By the above argument $\{0\} + \{0\} = \{0\}$, i.e. $\{0\}$ is its own inverse. Suppose $U \neq \{0\}$ is a subspace of V. Then U has some nonzero element u. If W is any other subspace of V, then $0 \in W$. Hence $0 \neq u = u + 0 \in U + W$. Therefore $U + W \neq \{0\}$. So there cannot exist a subspace W such that $U + W = \{0\}$.

4. (10 pts each) Let V be a vector space. Prove the following.
   (a) The additive identity of V is unique.

   See Proposition 1.2 in your textbook.

   (b) For all $v \in V$, $(-1)v = -v$.

   See Proposition 1.6 in your textbook.

5. (10 pts) Recall that the trace of a square matrix is the sum of its diagonal entries. E.g.

$$\operatorname{tr}\begin{pmatrix} -1 & 5 \\ 0 & 3 \end{pmatrix} = -1 + 3 = 2.$$

Let $V = M_2(\mathbb{C})$ be the vector space of 2 by 2 matrices over the field $\mathbb{C}$. We verified in class that V was indeed a vector space. Let

$$U = \{A \in V \mid \operatorname{tr}(A) = 0\}.$$

That is U is the set of those 2 by 2 matrices whose trace is 0. Is U a subspace of V? If so, prove it is. If not, explain why not.

U is indeed a subspace of V. First note that

$$\operatorname{tr}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \implies \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in U.$$

Now, suppose $A, B \in U$. Then $\operatorname{tr}(A) = \operatorname{tr}(B) = 0$. So

$$\begin{aligned} \operatorname{tr}(A+B) &= \operatorname{tr}\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix} \\ &= a_{11}+b_{11}+a_{22}+b_{22} = \underbrace{a_{11}+a_{22}}_{0} + \underbrace{b_{11}+b_{22}}_{0} = 0. \end{aligned}$$

This shows $A + B \in U$. Now, let $\alpha \in \mathbb{C}$ and $A \in U$. Then $\operatorname{tr}(A) = 0$. So

$$\operatorname{tr}(\alpha A) = \operatorname{tr}\begin{pmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{pmatrix} = \alpha a_{11} + \alpha a_{22} = \alpha(\underbrace{a_{11}+a_{22}}_{0}) = 0 \implies \alpha A \in U.$$

Hence U is a subspace of V.

6. (10 pts) **Extra credit problem.** We showed in class that the set V of differentiable functions $\mathbb{R} \to \mathbb{R}$ forms a vector space over the field $\mathbb{R}$. Consider the first-order homogeneous linear differential equation

$$f'(x) - 3e^x f(x) = 0.$$

Let U be the set of all $\mathbb{R} \to \mathbb{R}$ functions which satisfy this differential equation. Prove that U is a subspace of V. (Hint: you don't need to solve the differential equation in order to do this.)

First, notice that the zero function $f(x) = 0$ satisfies this differential equation:

$$f'(x) - 3e^x f(x) = 0 - 3e^x 0 = 0.$$

So U contains 0. Now, suppose $f, g \in U$. Hence

$$f'(x) - 3e^x f(x) = 0 \text{ and } g'(x) - 3e^x g(x) = 0.$$

So

$$\begin{aligned} \frac{d}{dx}(f+g) - 3e^x(f+g)(x) &= f'(x) + g'(x) - 3e^x(f(x) + g(x)) \\ &= f'(x) - 3e^x f(x) + g'(x) - 3e^x g(x) \\ &= 0 + 0 = 0. \end{aligned}$$

Hence $f + g \in U$. Finally, let $\alpha \in \mathbb{R}$ and $f \in U$. Then

$$\frac{d}{dx}(\alpha f) - 3e^x(\alpha f)(x) = \alpha f'(x) - 3e^x \alpha f(x) = \alpha(f'(x) - 3e^x f(x)) = 0.$$

Hence $\alpha f \in U$. Therefore U is a subspace of V.