Math 524 Exam 2 Solutions

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1. (10 pts) Let F be a field. Prove that F^∞ is an infinite dimensional vector space.

   Suppose F^∞ is finite dimensional. Let $n = \dim(F^\infty)$.
   Let v_i be the sequence whose elements are all 0 except the i-th element which is 1. That is
   $$v_i = (0, \ldots, 0, 1, 0, \ldots).$$
   Now, let $S = (v_1, v_2, \ldots, v_{n+1})$. Notice that S_n is linearly independent since
   $$0 = \sum_{i=1}^{n+1} \alpha_i v_i = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1}, 0 \ldots)$$
   only if $\alpha_i = 0$ for all $1 \le i \le n+1$. But this would contradict Theorem 2.6 which says that no linearly independent set can be longer than the dimension.

2. (10 pts) Suppose that U and W are both 5-dimensional subspaces of $\mathbb{R}^9$. Prove that $U \cap W \neq \{0\}$.

   Since $U, W \subseteq \mathbb{R}^9$, $U + W$ is a subspace of $\mathbb{R}^9$. So $\dim(U+W) \le \dim(\mathbb{R}^9) = 9$. By Theorem 2.18
   $$\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U+W) = 10 - \dim(U+W) \ge 1.$$
   Hence $U \cap W \neq \{0\}$.

3. (10 pts) Let V be a vector space. Prove that if the list $(v_1, \ldots, v_n)$ is linearly independent in V, then so is the list
   $$(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$$
   obtained by subtracting from each vector (except the last one) the following vector.

   Let
   $$\begin{aligned} 0 &= \alpha_1(v_1 - v_2) + \alpha_2(v_2 - v_3) + \cdots + \alpha_{n-1}(v_{n-1} - v_n) + \alpha_n v_n \\ &= \alpha_1 v_1 + (\alpha_2 - \alpha_1)v_2 + \cdots + (\alpha_n - \alpha_{n-1})v_n \end{aligned}$$
   Since $(v_1, \ldots, v_n)$ is linearly independent, $\alpha_1 = 0$, $\alpha_2 - \alpha_1 = 0$, etc. Hence $\alpha_i = 0$ for all $1 \le i \le n$. This is exactly what we wanted to prove.

4. Let $n \in \mathbb{Z}^{\ge 0}$ and $V = \mathcal{P}_n(\mathbb{R})$. For $a \in \mathbb{R}$, define the map $T_a : V \to \mathbb{R}$ by $T(p) = p(a)$. For example $T_3(x^2 - 7) = 3^2 - 7 = 2$. This is called the evaluation map at a.
   (a) (6 pts) Prove that T_a is a linear map.

   Let $p, q \in V$ and $\alpha \in \mathbb{R}$. Then
   $$T_a(p+q) = (p+q)(a) = p(a) + q(a) = T_a(p) + T_a(q),$$
   and
   $$T_a(\alpha p) = (\alpha p)(a) = \alpha p(a) = \alpha T_a(p).$$
   Hence T_a is a linear map.

(b) (4 pts) Find the range(T_a). Is T_a surjective?

If $b \in \mathbb{R}$, then the constant polynomial $p(x) = b$ yields $T_a(p) = b$. This can be done for any $b \in \mathbb{R}$, so range(T_a) $= \mathbb{R}$. Indeed, T_a is surjective.

(c) (10 pts) What can you you say about dim(null(T_a))? Is T_a injective?

By Theorem 3.4,

$$\dim(\mathrm{null}(T_a)) = \dim(V) - \dim(\mathrm{range}(T_a)) = n + 1 - 1 = n.$$

So if $n > 0$ then dim(null(T_a)) > 0, so T_a is not injective. If $n = 0$, then T_a is injective. Indeed $P_0(\mathbb{R}) = \mathbb{R}$ and $T_a : \mathbb{R} \to \mathbb{R}$ is just the identity map in this case.

5. (10 pts) Let V and W be vector spaces over the same field F and $T = \mathcal{L}(V, W)$. Prove that T is injective if and only if null(T) $= \{0\}$.

See Proposition 3.2 in your textbook.

6. (10 pts) **Extra credit problem.** Let T_a be as in problem 4. Find a basis for null(T_a). (Hint: if p is a polynomial then $p(a) = 0$ iff $p(x) = (x - a)q(x)$ for some polynomial q.)

Notice that $0 = T_a(p) = p(a)$ iff $p(x) = (x - a)q(x)$ for some polynomial q. We already know that dim(null(T_a)) $= n$, so if we can come up with a linearly independent list of n polynomials of the form $p_i(x) = (x - a)q_i(x)$, then we have a basis for null(T_a) (by Prop 2.17). One such list is

$$((x - a), (x - a)x, (x - a)x^2, \ldots, (x - a)x^{n-1}).$$

To see that this is linearly independent, let

$$\begin{aligned} 0 &= \alpha_1(x - a) + \alpha_2(x - a)x + \cdots + \alpha_n(x - a)x^{n-1} \\ &= (x - a)(\alpha_1 + \alpha_2 x + \cdots + \alpha_{n-1}x^{n-1}). \end{aligned}$$

Notice that this is so exactly when

$$0 = \alpha_1 + \alpha_2 x + \cdots + \alpha_{n-1}x^{n-1}.$$

But this holds only when $\alpha_i = 0$ for all $1 \le i \le n - 1$.

Hence

$$((x - a), (x - a)x, (x - a)x^2, \ldots, (x - a)x^{n-1})$$

is a basis of null(T_a).