Math 524 Final Exam Solutions

May 17, 2011

1. (10 pts) Give an example of a nonempty subset U of $\mathbb{R}^2$ (over the field $\mathbb{R}$) such that U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}^2$.

   Let U be the union of the x-axis and the y-axis:
   $$U = \{(x,0) \mid x \in \mathbb{R}\} \cup \{(0,y) \mid y \in \mathbb{R}\}.$$
   Let $\alpha \in \mathbb{R}$. If $u \in U$ then either $u = (x,0)$ for some $x \in \mathbb{R}$ or $u = (0,y)$ for some $y \in \mathbb{R}$. In the first case, $\alpha u = (\alpha x, 0) \in U$. In the second case, $\alpha u = (0, \alpha y) \in U$. So U is closed under scalar multiplication.

   But U is not closed under addition. E.g. $(1,0),(0,1) \in U$, but $(1,1) \notin U$. Hence U is not a subspace of $\mathbb{R}^2$.

2. (10 pts) Suppose $(v_1,\ldots,v_n)$ is linearly independent in V and $w \in V$. Prove that if $(v_1 + w,\ldots,v_n + w)$ is linearly dependent, then $w \in \operatorname{span}(v_1,\ldots,v_n)$.

   Suppose $(v_1 + w,\ldots,v_n + w)$ is linearly dependent. Then there exist scalars $\alpha_1,\ldots,\alpha_n$ not all 0 such that
   $$\alpha_1(v_1 + w) + \cdots + \alpha_n(v_n + w) = 0.$$
   Hence
   $$-\left(\sum_{i=1}^{n} \alpha_i\right) w = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$
   If $\sum \alpha_i = 0$, then
   $$0 = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$
   But this contradicts the linear independence of $(v_1,\ldots,v_n)$ since we know at least one of the $\alpha_i \neq 0$. Therefore $\sum \alpha_i \neq 0$ and
   $$w = \frac{-\alpha_1}{\sum \alpha_i} v_1 + \cdots + \frac{-\alpha_n}{\sum \alpha_i} v_n \in \operatorname{span}(v_1,\ldots,v_n).$$

3. (10 pts) Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that
   $$f(av) = af(v)$$
   for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$, but f is not linear.

   Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x,y) = \sqrt[3]{x^2 y}$. Then
   $$f(a(x,y)) = f((ax,ay)) = \sqrt[3]{(ax)^2(ay)} = a\sqrt[3]{x^2 y} = af(x,y).$$
   But f is not additive. E.g.
   $$f(1,0) + f(0,1) = 0 + 0 = 0$$
   and
   $$f((1,0) + (0,1)) = f(1,1) = 1.$$

4. (10 pts) Prove that every polynomial with odd degree and real coefficients has a real root.

   Let p be such a polynomial. By Theorem 4.14, p can be factored as a product of linear and quadratic factors. If all these factors were quadratic, the degree of p would have to be even. Therefore there is at least one linear factor $x - \lambda$, and this λ is a real root of p.

5. (8 pts each) Let V and W be a vector spaces over the field F and $T \in \mathcal{L}(V, W)$. Let $U_1, U_2 \subseteq V$ be subspaces. Prove or disprove the following statements.

(a) $T(U_1 + U_2) = T(U_1) + T(U_2)$

This is true. Let $v \in T(U_1 + U_2)$. Then $v = T(u)$ for some $u \in U_1 + U_2$. Such a u must be of the form $u = u_1 + u_2$ for some $u_1 \in U_1$ and $u_2 \in U_2$. Hence

$$v = T(u) = T(u_1 + u_2) = T(u_1) + T(u_2) \in T(U_1) + T(U_2).$$

So $T(U_1 + U_2) \subseteq T(U_1) + T(U_2)$.
Now let $v = T(U_1) + T(U_2)$. So there exist $u_1 \in U_1$ and $u_2 \in U_2$ such that $v = T(u_1)+T(u_2)$. Then $v = T(u_1)+T(u_2) = T(u_1+u_2) \in T(U_1+U_2)$. So $T(U_1)+T(U_2) \subseteq T(U_1 + U_2)$. Now we know $T(U_1 + U_2) = T(U_1) + T(U_2)$.

(b) $T(U_1 \oplus U_2) = T(U_1) \oplus T(U_2)$

This is false. We already know $T(U_1 + U_2) = T(U_1) + T(U_2)$ from part (a), but we will show by a counterexample that the directness of the first sum does not imply the directness of the second sum.
Let $V = W = \mathbb{R}^2$ over $\mathbb{R}$ and $T(x, y) = (x + y, 0)$. Let U_1 be the x-axis and U_2 be the y-axis. First, notice that

$$U_1 + U_2 = \{(x, 0) + (0, y) \mid x, y \in \mathbb{R}\} = \{(x, y) \mid x, y \in \mathbb{R}\} = \mathbb{R}^2.$$

Next, $(a, b) = (x, 0) + (0, y)$ implies $x = a$ and $y = b$, so there is exactly one way to write every vector in $U_1 + U_2 = \mathbb{R}^2$ as a sum $u_1 + u_2$ with $u_1 \in U_1$ and $u_2 \in U_2$. Hence the sum above is direct. Finally,

$$T(U_1) = \{T(x, 0) \mid x \in \mathbb{R}\} = \{(x, 0) \mid x \in \mathbb{R}\} = U_1$$
$$T(U_2) = \{T(0, y) \mid y \in \mathbb{R}\} = \{(y, 0) \mid y \in \mathbb{R}\} = U_1.$$

Therefore $T(U_1) + T(U_2) = U_1 + U_1 = U_1$, but this sum is not direct. E.g. $(0, 0) = (0, 0) + (0, 0) = (1, 0) + (-1, 0)$.

6. (10 pts) Let V and W be vector spaces and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if $\text{null}(T) = \{0\}$.

See Proposition 3.2 in your textbook. Or those of you who have taken Math 521B, should recognize this as Proposition 3.7.4(b) in Beachy and Blair.

7. (14 pts) Let V be a finite-dimensional vector space over the complex numbers and $T \in \mathcal{L}(V)$. Show that T has an eigenvector (and eigenvalue of course).

See Theorem 5.10 in your textbook.

8. (10 pts) **Extra credit problem.** Prove that the finite-dimensionality of V is a necessary condition in problem 7. (Hint: find an infinite-dimensional complex vector space V and $T \in \mathcal{L}(V)$ such that T has no eigenvector.)

Let $V = \mathbb{R}^\infty$ and T the right shift operator

$$T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots).$$

Then if

$$\lambda(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$$

then

$$\lambda x_1 = 0, \quad \lambda x_2 = x_1, \quad \lambda x_3 = x_2, \quad \ldots.$$

Now if $\lambda = 0$ then $0 = x_1 = x_2 = \cdots$. But an eigenvector must be nonzero. If $\lambda \neq 0$ then $x_1 = 0$ hence $x_2 = 0$, etc. That still cannot be an eigenvector. Therefore T does not have any eigenvector/eigenvalue.

Another example, following essentially the same idea, would be $V = \mathbb{R}[x]$ and $T(p) = xp$.

9. (10 pts) **Extra credit problem.** Let V be a finite-dimensional vector space and $S, T \in \mathcal{L}(V)$. Prove that 0 is an eigenvalue of ST if and only if 0 is an eigenvalue of S or T.

Suppose 0 is an eigenvalue of ST. Then there is a $v \neq 0$ such that $ST(v) = 0$. If $T(v) = 0$ then v is also an eigenvector of T with eigenvalue 0. If $T(v) \neq 0$ then $T(v)$ is an eigenvector of S with eigenvalue 0.

Conversely, suppose 0 is an eigenvalue of S or T. If 0 is an eigenvalue of T then $T(v) = 0$ for some $v \neq 0$. So $ST(v) = S(0) = 0$. That is 0 is an eigenvalue of ST. If 0 is not an eigenvalue of T then it must be an eigenvalue of S. So $S(v) = 0$ for some $v \neq 0$. Since 0 is not an eigenvalue of T, $\text{null}(T) = \{0\}$. So T is injective (by Prop 3.2). But T must then be surjective too (by Theorem 3.21). So $v = T(w)$ for some w. Since $v \neq 0$, also $w \neq 0$. Now $ST(w) = S(v) = 0$, so 0 is an eigenvalue of ST.