Math 524 Exam 1 Solutions

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1. (5 pts) Let F be any field. Determine if the following subset of F^3 is a subspace of F^3:
$$\{(x_1, x_2, x_3) \in F^3 : x_1x_2x_3 = 0\}.$$

   Call the above set U. It is not a subspace of F^3 because it is not closed under addition. For example, let $u_1 = (1,1,0)$ and $u_2 = (0,0,1)$. It is clear that $u_1, u_2 \in U$. But $u_1 + u_2 = (1,1,1) \notin U$.

2. (10 pts) Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}^2$.

   Let $U = \{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$. Notice that U is actually the union of the x-axis and y-axis. Obviously, $(0,0) \in U$, so U is not empty. Let $u = (x,y)$ be any vector in U and $\alpha \in \mathbb{R}$. Then $\alpha u = (\alpha x, \alpha y)$. Since $u \in U$, $xy = 0$. So $(\alpha x)(\alpha y) = \alpha^2 xy = 0$. Hence $\alpha v \in U$. So U is closed under scalar multiplication. Now $(1,0),(0,1) \in U$, but $(1,0)+(0,1) = (1,1) \notin U$, so U is not closed under addition.

3. (10 pts) Let V be a vector space. Suppose that U is a subspace of V. What is $U + U$?

   We will show that $U + U = U$. First, let $u \in U$. Then $u = u + 0 \in U + U$. This is true for all $u \in U$, so $U \subseteq U + U$. Now, let $u \in U + U$. Then $u = u_1 + u_2$ for some $u_1, u_2 \in U$. But U is a subspace, so U is closed under addition. Hence $u \in U$. This is true for all $u \in U + U$, so $U + U \subseteq U$. Therefore $U + U = U$.

4. (10 pts) Let V be a vector space over the field F. Prove that $a \cdot 0 = 0$ for all $a \in F$.

   See Proposition 1.5 in your textbook.

5. (a) (3 pts) Let V be a vector space and $U_1, U_2, \ldots, U_n$ be subspaces of V. Define $U_1 + U_2 + \cdots + U_n$.

   $$U_1 + U_2 + \cdots + U_n = \{u_1 + u_2 + \cdots + u_n \mid u_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n\}$$

   (b) (3 pts) Define what it means for $U_1 + U_2 + \cdots + U_n$ to be a direct sum.

   The sum $U_1 + U_2 + \cdots + U_n$ is direct if every $u \in U_1 + U_2 + \cdots + U_n$ can be written as $u = u_1 + u_2 + \cdots + u_n$ uniquely.

   (c) (9 pts) Let $V = \mathbb{R}^3$ over $\mathbb{R}$ and $U_1 = \{(x,y,z) \mid x+y+z = 0\}$. Find a subspace $U_2 \subseteq V$ such that $V = U_1 \oplus U_2$. (Obviously, you need to prove that $V = U_1 \oplus U_2$.)

   Let $U_2 = \{(x,0,0) \mid x \in \mathbb{R}$. That is U_2 is the x-axis. Since U_1 and U_2 are both subspaces of V, $U_1 + U_2 \subseteq V$. Now, let $v = (x,y,z)$ be any vector in V. Let $u_1 = (-y-z, y, z)$ and $u_2 = (y+z, 0, 0)$. Notice that $u_1 \in U_1$, $u_2 \in U_2$, and $v = u_1 + u_2$. So $V \subseteq U_1 + U_2$. Hence $V = U_1 + U_2$.
   By Proposition 1.9, if $U_1 \cap U_2 = \{0\}$ then $U_1 + U_2$ is direct. Let $v = (x,y,z)$ be in $U_1 \cap U_2$. Then $v \in U_2$, so $y = z = 0$. But $v \in U_1$, so $x = x + \underbrace{y+z}_{0} = 0$ too. So $v = 0$.

   Hence $U_1 \cap U_2 = \{0\}$.

6. (10 pts) **Extra credit problem.** Let V be a vector space and $U_1, U_2, \ldots, U_n$ be subspaces of V. Prove that $U = U_1 + U_2 + \cdots + U_n$ is the smallest subspace of V such that $U_i \subseteq U$ for all $1 \leq i \leq n$. I.e. you need to prove that if W is a subspace of V such that $U_i \subseteq W$ for all $1 \leq i \leq n$ then $U \subseteq W$.

   Let W be a subspace of V such that $U_i \subseteq W$ for all $1 \leq i \leq n$ then $U \subseteq W$. Let $u \in U_1 + \cdots + U_n$. Then $u = u_1 + \cdots + u_n$ for some $u_i \in U_i$ for $1 \leq i \leq n$. Since $u_i \in U_i$ and $U_i \subseteq W$, then $u_i \in W$ for all i. But W is a subspace, so it is closed under addition. Hence $u = u_1 + \cdots + u_n \in W$. This is true for all $u \in U$, so $U \subseteq W$.