Math 524 Exam 2 Solutions

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1. (10 pts) Suppose that V is a finite dimensional vector space with $\dim(V) = n$. Prove that there exist 1-dimensional subspaces $U_1, \dots, U_n$ of V such that
$$V = U_1 \oplus \cdots \oplus U_n.$$

   Since V is finite-dimensional, it has a basis $(v_1, \dots, v_n)$ (Corollary 2.11). Let
$$U_i = \operatorname{span}(v_i) = \{\alpha v_i \mid \alpha \in F\}.$$
   Then (v_i) is a basis of U_i, so U_i is one-dimensional. Since $U_i \subseteq V$ for all i, $U_1 + \cdots + U_n \subseteq V$. Now, if $v \in V$, then there exist $\alpha_1, \dots, \alpha_n \in F$ such that
$$v = \underbrace{\alpha_1 v_1}_{\in U_1} + \underbrace{\alpha_2 v_2}_{\in U_2} + \cdots + \underbrace{\alpha_n v_n}_{\in U_n} \in U_1 + \cdots + U_n.$$
   So $V \subseteq U_1 + \cdots + U_n$. Hence $V = U_1 + \cdots + U_n$.

   Since $(v_1, \dots, v_n)$ is a basis of V, this is also the only way to express v as a sum of scalar multiples of $v_1, \dots, v_n$. Hence there is no other way to write v as a sum of vectors $u_1 + \cdots + u_n$ with $u_i \in U_i$ for all i. Therefore the sum $U_1 + \cdots + U_n$ is direct and $V = U_1 \oplus \cdots \oplus U_n$.

2. (10 pts) Suppose that V is finite dimensional and U is a subspace of V such that $\dim(U) = \dim(V)$. Prove that $U = V$.

   Since U is a subspace of a finite dimensional vector space, it is finite dimensional (Prop 2.7). So U has a basis $(u_1, \dots, u_n)$. This can be extended to a basis of V (Theorem 2.12). But since $\dim(V) = \dim(U) = n$, the extension must be just $(u_1, \dots, u_n)$. So every $v \in V$ is a linear combination of $(u_1, \dots, u_n)$. Hence every $v \in V$ is in U. So $V \subseteq U$. Since U is a subspace of V, $U \subseteq V$. Hence $U = V$.

3. (10 pts) Suppose that $T \in \mathcal{L}(V, W)$ is injective and $(v_1, \dots, v_n)$ is linearly independent in V. Prove that $(T(v_1), \dots, T(v_n))$ is linearly independent in W.

   Let $\alpha_1, \dots, \alpha_n \in F$ be such that
$$\sum_{i=1}^{n} \alpha_i T(v_i) = 0.$$
   We need to prove that $\alpha_1 = \cdots = \alpha_n = 0$. By linearity,
$$0 = \sum_{i=1}^{n} \alpha_i T(v_i) = T\left(\sum_{i=1}^{n} \alpha_i v_i\right).$$
   Since T is injective, the only vector in its null space is 0. So
$$\sum_{i=1}^{n} \alpha_i v_i = 0.$$
   Since $(v_1, \dots, v_n)$ is linearly independent, this happens only if $\alpha_1 = \cdots = \alpha_n = 0$.

4. (a) (3 pts) Let V be a vector space over the field F and $v_1, \dots, v_n \in V$. Define the span of $v_1, \dots, v_n$.

The span of $v_1, \dots, v_n \in V$ is the set of all linear combinations of $v_1, \dots, v_n \in V$. I.e.

$$\text{span}(v_1, \dots, v_n) = \{\sum_{i=1}^{n} \alpha_i v_i \mid \alpha_i \in F \forall 1 \leq i \leq n\}.$$

(b) (2 pts) Define what a finite dimensional vector space is.

A vector space is finite dimensional if is spanned by a finite list (or set) of vectors.

(c) (10 pts) Prove that every subspace of a finite dimensional vector space is finite dimensional.

See Proposition 2.7 in your textbook.

5. (a) (3 pts) Let $f : A \to B$ be a map. Define what it means for f to be injective.

The map $f : A \to B$ is injective if for all $x, y \in A$, $f(x) = f(y)$ implies $x = y$

(b) (2 pts) Let V and W be vector spaces over the same field F and let T be a linear map $T \in \mathcal{L}(V, W)$. Define the null space of T.

The null space of the linear map $T \in \mathcal{L}(V, W)$ is

$$\text{null}(T) = \{v \in V \mid T(v) = 0\}.$$

(c) (10 pts) Prove that a linear map $T \in \mathcal{L}(V, W)$ is injective if and only if its null space is $\{0\}$.

See Proposition 3.2 in your textbook.

6. (5 pts each) **Extra credit problem.** Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called even if $f(-x) = f(x)$ for all $x \in \mathbb{R}$ and odd if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

(a) Let $V = \mathbb{R}[x]$ over the field $\mathbb{R}$. Is the subset $W \subseteq V$ which consists of all those polynomials in V that are odd functions a subspace of V?

Let p and q be odd polynomials. Then

$$(p + q)(-x) = p(-x) + q(-x) = -p(x) - q(x) = -(p + q)(x).$$

This shows $p + q$ is odd. So W is closed under addition.
Now, let p be an odd polynomial and $\alpha \in F$. Then

$$(\alpha p)(-x) = \alpha p(-x) = \alpha(-p(x)) = -\alpha p(x) = -(\alpha p)(x).$$

This shows αp is odd. So W is closed under scalar multiplication.
Finally, observe that $0(-x) = 0 = -0(x)$. So the zero polynomial is in W. Hence W is a subspace of V.

(b) Now, let $V = \mathcal{P}_5(\mathbb{R})$ be the vector space of polynomials of degree less than or equal to 5 over the field $\mathbb{R}$. Let $U \subseteq V$ consist of all of all those polynomials in V that are even functions, i.e. satisfy $p(-x) = p(x)$ for all $x \in \mathbb{R}$. Prove that U is a subspace of V and find its dimension.

The proof that U is a subspace is analogous to the one in part (a).

Now, let $p(x) = a_n x^n + \cdots + a_0$ be any polynomial. We may assume without loss of generality that n is even because we do not have to have $a_n \neq 0$. If p is even, then $p(-x) = p(x)$ implies

$$\begin{aligned} 0 &= p(x) - p(-x) \\ &= a_n x^n + \cdots + a_0 - (a_n(-x)^n + \cdots + a_0) \\ &= 2a_{n-1}x^{n-1} + 2a_{n-3}x^{n-3} \cdots + 2a_1 x \end{aligned}$$

Hence $a_1 = a_3 = \cdots = a_{n-1} = 0$. That is p can have only terms with even exponents. Therefore if $p \in U$, then $p(x) = ax^4 + bx^2 + c$. So $(1, x^2, x^4)$ spans U. It is clear that $ax^4 + bx^2 + c = 0$ only if $a = b = c = 0$. So $(1, x^2, x^4)$ is also linearly independent. Hence $(1, x^2, x^4)$ is a basis of U and $\dim(U) = 3$.