Math 524 Final Exam Solutions

May 14, 2013

1. (10 pts) Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that
$$V = U_1 \oplus W \qquad \text{and} \qquad V = U_2 \oplus W$$
then $U_1 = U_2$.

Here is a counterexample. Let $V = \mathbb{R}^2$, U_1 the x-axis, U_2 the y-axis, and $W = \{(x,x) \mid x \in \mathbb{R}\}$. We will show that $U_1 + W = V$ and $U_2 + W = V$.

Since $U_1, W \subseteq V$, $U_1 + W \subseteq V$. Let $v = (x,y)$ in V. Then
$$v = \underbrace{(x-y,0)}_{\in U_1} + \underbrace{(y,y)}_{\in U_2} \in U_1 + W.$$
So $V \subseteq U_1 + W$. Hence $U_1 + W = V$. That the sum is direct follows from $U_1 \cap W = \{(0,0)\}$.

An analogous argument shows that $V = U_2 \oplus W$. So $U_1 \oplus W = U_2 \oplus W$, but obviously $U_1 \neq U_2$.

2. (10 pts) Let U be the subspace of $\mathbb{R}^5$ defined by
$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$
Find a basis of U.

Notice that any vector in $u \in U$ is of the form $(3x_2, x_2, 7x_4, x_4, x_5)$ where $x_2, x_4, x_5 \in \mathbb{R}$. Therefore
$$u = x_2(3,1,0,0,0) + x_4(0,0,7,1,0) + x_5(0,0,0,0,1)$$
and
$$U = \operatorname{span}\left((3,1,0,0,0), (0,0,7,1,0), (0,0,0,0,1)\right).$$
Observe that
$$\begin{aligned}(0,0,0,0,0) &= \alpha_1(3,1,0,0,0) + \alpha_2(0,0,7,1,0) + \alpha_3(0,0,0,0,1)\\ &= (3\alpha_1, \alpha_1, 7\alpha_2, \alpha_2, \alpha_3)\end{aligned}$$
implies $\alpha_1 = \alpha_2 = \alpha_3 = 0$. So these vectors are linearly independent. Therefore they form a basis of U.

3. (10 pts) Suppose that $T \in \mathcal{L}(V)$ is such that every vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

We are asked to prove that there exists a scalar λ such that $T(v) = \lambda v$ for all $v \in V$. We already know that for every $v \in V$ there is a scalar λ_v such that $T(v) = \lambda_v v$. But we need to prove that these λ_v's are the same and do not depend on v.

If V is $\{0\}$ or V has only one nonzero vector in it, then the statement is obviously true.

So let $v \neq u$ be nonzero vectors in V. We know that there exist $\lambda_v, \lambda_u \in F$ such that $T(v) = \lambda_v v$ and $T(u) = \lambda_u u$. Suppose $\lambda_v \neq \lambda_u$. Now, we also know that there exists $\lambda_{v+u} \in F$ such that $T(v+u) = \lambda_{v+u}(v+u)$. But by linearity,
$$T(v+u) = T(v) + T(u) = \lambda_v v + \lambda_u u$$
Hence
$$\lambda_v v + \lambda_u u = \lambda_{v+u}(v+u) = \lambda_{v+u} v + \lambda_{v+u} u.$$
So
$$0 = (\lambda_{v+u} - \lambda_v)v + (\lambda_{v+u} - \lambda_u)u.$$

But eigenvectors that correspond to distinct eigenvalues are linearly independent. So the above equality can only hold if $\lambda_{v+u} - \lambda_v = 0$ and $\lambda_{v+u} - \lambda_u = 0$. Hence $\lambda_v = \lambda_{v+u} = \lambda_u$. But this contradicts $\lambda_v \neq \lambda_u$.

4. (10 pts) Let V be a finite dimensional vector space and $T \in \mathcal{L}(V, V)$. Prove that the following are equivalent:
   (a) T is bijective,
   (b) T is injective,
   (c) T is surjective.

   This is Theorem 3.21 in your textbook.

5. (10 pts) Let V and W be vector spaces such that $\dim(V) < \infty$. Let $T \in \mathcal{L}(V, W)$. Prove that the range of T is finite dimensional and

$$\dim(V) = \dim(\text{null}(T)) + \dim(\text{range}(T)).$$

   This is Theorem 3.4 in your textbook.

6. (10 pts) Prove that every linear operator on a finite dimensional complex vector space has an eigenvalue.

   This is Theorem 5.10 in your textbook.

7. (10 pts) **Extra credit problem.** Prove that $\dim_{\mathbb{Q}}(\mathbb{R}) = \infty$. (Hint: prove that you can build as long a linearly independent list as you want from the square roots of prime numbers.)

   If you follow the hint, you can argue that

$$(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \ldots)$$

   is as long a linearly independent list as you want. If $\mathbb{R}$ were finite dimensional over $\mathbb{Q}$ then it would have a finite spanning list and no linearly independent list could be longer than that spanning list. The trick here is to prove the linear independence of the list above. I will leave that challenge to you.

   Here is another approach, if you know about countable and uncountable sets. Suppose $\mathbb{R}$ is finite dimensional over $\mathbb{Q}$. Then $\mathbb{R}$ has a basis $(v_1, \ldots, v_n)$. Every vector in $\mathbb{R}$ can then be expressed uniquely as $\alpha_1 v_1 \cdots \alpha_n v_n$ where $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}$. But the number of elements in the set

$$\{\alpha_1 v_1 \cdots \alpha_n v_n \mid \alpha_1, \ldots, \alpha_n \in \mathbb{Q}\}$$

   is the same as in the set

$$\underbrace{\mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}}_{n \text{ times}} = \mathbb{Q}^n,$$

   which is countable because it is a finite Cartesian product of countable sets. But $\mathbb{R}$ is not countable.

   Finally, here is yet another approach if you know about transcendental numbers. A transcendental number is an irrational number that is not a root of any nonzero polynomial with rational coefficients. For example, π is a well-known example of a transcendental number. Hence

$$(1, \pi, \pi^2, \pi^3, \ldots)$$

   is a linearly independent list that you can make as long as you want. Hence $\mathbb{R}$ cannot be finite dimensional over $\mathbb{Q}$.