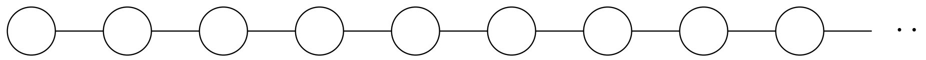
MCS 115 Final Exam

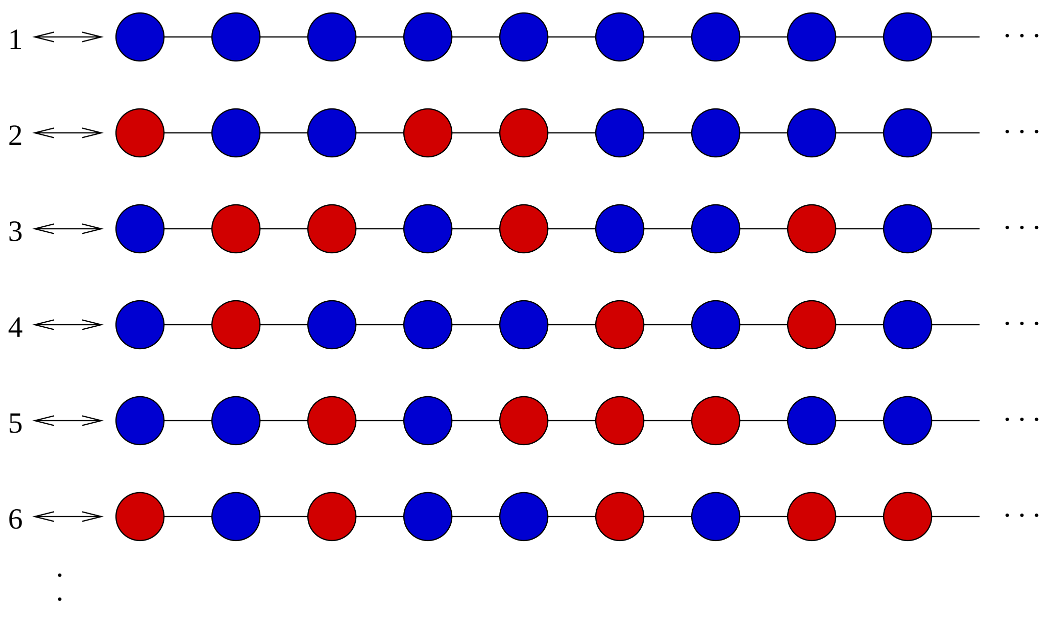
May 26, 2018

1. (10 pts) Consider the infinite collection of circles below:

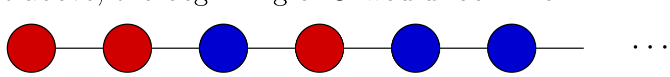


Suppose you have two markers, one red and one blue, and you color each circle one of the two colors. Show that the set of all possible circle colorings has a greater cardinality than the set of all natural numbers.

We need to show that there cannot be a one-to-one correspondence between the natural numbers and the set of all colored chains. Suppose that there were such a one-to-one correspondence. Then we could list the natural numbers in one column and the corresponding colored chains in the column next to it. E.g. the beginning of the list could look like this:

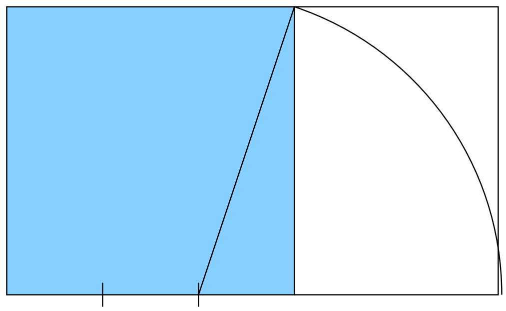


We will now show that no matter what, we can always construct another colored chain C that is not on the list. This will show that the above list cannot be a one-to-one correspondence between $\mathbb{N}$ and the set of <u>all</u> circle colorings. To construct such a colored chain C, we can use Cantor's diagonal argumenent. For ease of reference, I will refer to the colored chain that corresponds to the natural number n as C_n. So first, look at the first circle in C_1. If it is red, then color the first circle of C blue; and if it is blue, then color the first circle of C red. Now look at the second circle of C_2 and color the second circle of C the opposite color. And so on, color the n-th circle of C blue if the n-th circle of C_n is red; and color it red otherwise. E.g. for the list above, the beginning of C would look like:



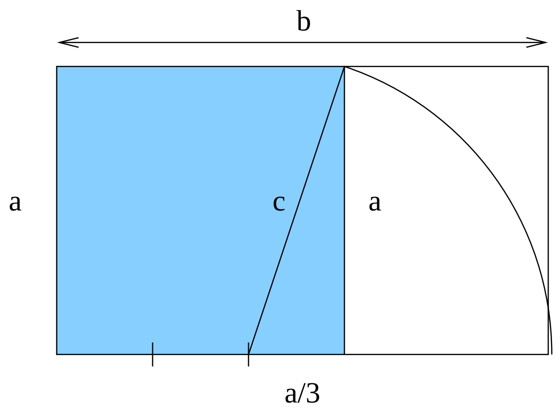
Now, notice that C is different from C_1 because their first circles are different colors. But C is also different from C_2 because their second circles are different colors. And so on, C is different from C_n because their n-th circles are different colors. Indeed, C cannot match any of the chains on the list above. Therefore, C is not on the list, so the list cannot be a one-to-one correspondence between $\mathbb{N}$ and all colored chains. This can be done for any list of colored chains, and hence no one-to-one correspondence can exist between $\mathbb{N}$ and all colored chains.

2. (10 pts) Let's make a rectangle somewhat like the Golden Rectangle. As before, start with a square; however, instead of cutting the base in half, cut it into thirds and draw the line from the upper right vertex of the square to the point on the base that is one-third of the way from the right bottom vertex. Now use this new line segment as the radius of the circle, and continue as we did in the construction of the Golden Rectangle. This produces a new, longer rectangle, as shown in the diagram.



   (a) What is the ratio of the base to the height of this rectangle (that is, what is base/height for this new rectangle)?

   Let a be the side of square $ABCD$ and b the longer side of rectangle $ABEF$:



   Notice that $\triangle GCD$ is a right triangle whose legs are a and $a/3$. Therefore the hypotenuse is

$$c = \sqrt{a^2 + \left(\frac{a}{3}\right)^2} = \sqrt{a^2 + \frac{a^2}{9}} = \sqrt{\frac{9a^2}{9} + \frac{a^2}{9}} = \sqrt{\frac{10a^2}{9}} = \sqrt{\frac{10}{9}}a = \frac{\sqrt{10}}{3}a.$$

   This is also the radius of the circle, so $GE = \frac{\sqrt{10}}{3}a$. Hence

$$b = \frac{2}{3}a + \frac{\sqrt{10}}{3}a = \frac{2+\sqrt{10}}{3}a.$$

   The ratio of base to height is

$$\frac{\frac{2+\sqrt{10}}{3}a}{a} = \frac{2+\sqrt{10}}{3}.$$

   Note that not surprisingly, this is not the golden ratio, although that is not immediately obvious. It could be that $\frac{2+\sqrt{10}}{3}$ and $\frac{1+\sqrt{5}}{2}$ are two different ways of writing the same number. The quickest, although not very elegant, way to see that they are not the same is to use a calculator to calculate

$$\frac{2+\sqrt{10}}{3} \approx 1.7208 \quad \text{and} \quad \frac{1+\sqrt{5}}{2} \approx 1.6180$$

which are obviously different.

If you would like a more exact argument, here is one. We will show that $\frac{2+\sqrt{10}}{3} - \frac{1+\sqrt{5}}{2}$ is not 0, hence $\frac{2+\sqrt{10}}{3} \neq \frac{1+\sqrt{5}}{2}$.

$$\begin{aligned}\frac{2+\sqrt{10}}{3} - \frac{1+\sqrt{5}}{2} &= \frac{4+2\sqrt{10}}{6} - \frac{3+3\sqrt{5}}{6}\\ &= \frac{4+2\sqrt{10}-3-3\sqrt{5}}{6}\\ &= \frac{1+2\sqrt{10}-3\sqrt{5}}{6}.\end{aligned}$$

For this fraction to be 0, the numerator $1+2\sqrt{10}-3\sqrt{5}$ would have to be 0. If $1+2\sqrt{10}-3\sqrt{5}=0$ then $1+2\sqrt{10}=3\sqrt{5}$. But then we can square both sides and get

$$\begin{aligned}\left(1+2\sqrt{10}\right)^2 &= \left(3\sqrt{5}\right)^2 \\ 1+4\sqrt{10}+4\cdot 10 &= 9\cdot 5 \\ 41+4\sqrt{10} &= 45 && \text{subtract 41 from both sides}\\ 4\sqrt{10} &= 4 && \text{divide both sides by 4}\\ \sqrt{10} &= 1\end{aligned}$$

But that is not true, $1^2 = 1 \neq 10$, so $\sqrt{10} \neq 1$. Therefore $1+2\sqrt{10}-3\sqrt{5} \neq 0$ and hence

$$\frac{2+\sqrt{10}}{3} - \frac{1+\sqrt{5}}{2} \neq 0.$$

That is a bit of a long argument, but the calculations are quite standard.

(b) Now remove the largest square possible from this new rectangle and notice that we are left with another rectangle. Are the proportions of the base/height of this smaller rectangle the same as the proportions of the big rectangle?

One side of that rectangle is a. The other is

$$CE = GE - GC = \frac{\sqrt{10}}{3}a - \frac{a}{3} = \frac{\sqrt{10}-1}{3}a.$$

Note that this is smaller than a since $\sqrt{10} < 4$, so $\sqrt{10}-1 < 3$, so $\frac{\sqrt{10}-1}{3} < 1$. So the base to height ratio of rectangle $CDFE$ is

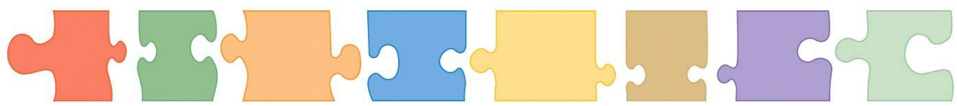
$$\frac{a}{\frac{\sqrt{10}-1}{3}a} = \frac{1}{\frac{\sqrt{10}-1}{3}} = \frac{3}{\sqrt{10}-1} \approx 1.3874,$$

which is different from the base to height ratio of the larger rectangle.

If you do not like decimal approximations, you can give a similar argument to the one in part (a) for

$$\frac{3}{\sqrt{10}-1} - \frac{2+\sqrt{10}}{3} \neq 0.$$

3. (10 pts) Suppose you have a collection of jigsaw pieces as shown below. They can be put together to form a strip. Can they be assembled into a Möbius band? Explain why or why not. Look for a mathematical reason for your answer, not a physical one such as the pieces will not link tightly enough, since you do not actually know what material the pieces are made of.



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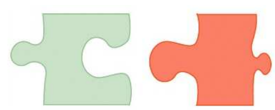
To assemble these jigsaw pieces into a Möbius band, we would have to give the strip a half-twist and then close it up into a band. This would mean that at one of the connections, one of the jigsaw pieces is now upside down, while the other is not. (Note this does not have to be between the first and the last pieces.) But then they can no longer be connected because the connecting parts of any two adjacent jigsaw pieces lack the symmetry that would be necessary for them to still connect this way. E.g. here is what this would look like with the first and last jigsaw pieces, which would have to be adjacent in a Möbius band:



The problem would be similar between any two adjacent pieces.

4. (10 pts) Find the remainder of 4^{2018} when divided by 11. Be sure to justify your answer.

Note that you had a few such problems in your online homework. We have

$$
\begin{aligned}
4^1 &= 4 \equiv 4 \pmod{11}\\
4^2 &= 16 = 11 + 5 \equiv 5 \pmod{11}\\
4^3 &= 4 \cdot 5 = 20 = 11 + 9 \equiv 9 \pmod{11}\\
4^4 &= 4 \cdot 9 = 36 = 3 \cdot 11 + 3 \equiv 3 \pmod{11}\\
4^5 &= 4 \cdot 3 = 12 = 11 + 1 \equiv 1 \pmod{11}\\
4^6 &= 4 \cdot 1 = 4 \equiv 4 \pmod{11}\\
4^7 &= 4 \cdot 4 = 16 = 11 + 5 \equiv 5 \pmod{11}\\
4^8 &= 4 \cdot 5 = 20 = 11 + 9 \equiv 9 \pmod{11}\\
4^9 &= 4 \cdot 9 = 36 = 3 \cdot 11 + 3 \equiv 3 \pmod{11}\\
4^{10} &= 4 \cdot 3 = 12 = 11 + 1 \equiv 1 \pmod{11}\\
&\vdots
\end{aligned}
$$

The remainders repeat from now on, which is because we keep doing the same calculations over and over again. In fact, we do not need to know that they repeat. All we need to know is that $4^5 \equiv 1 \pmod{11}$ to write

$$4^{2018} = 4^{403 \cdot 5 + 3} = \underbrace{4^5 \cdot 4^5 \cdots 4^5}_{403 \text{ times}} \cdot 5^3 \equiv 1 \cdot 1 \cdots 1 \cdot 9 \equiv 9 \pmod{11}.$$

So the remainder of 4^{2018} when divided by 11 is 9.

5. (10 pts) Prove that **any** connected, planar graph has Euler characteristic $V - E + R = 2$, where V is the number of vertices, E is the number of edges, and R is the number of regions the graph divides the plane into when drawn so that its edges only meet at the vertices. **Any** means your argument needs to be general enough to apply to every connected, planar graph that anyone could dream up. So counting V, E, and R for a particular graph of your choice, or two, or three, or even a million graphs is not sufficient evidence that any connected, planar graph must have the same property. (Hint: Build the graph one edge at a time while keeping track of what happens to $V - E + R$.)

Let G be any connected, planar graph. If G has no edges, then G can only have one vertex, otherwise G would be disconnected. In that case, $V = 1$ and $R = 1$ as one vertex cannot divide the plane into regions. And surely, $V - E + R = 1 - 0 + 1 = 2$ in this case.

Otherwise, we will build up G one edge at a time. Start with any edge and add it along with the two vertices at its ends. At this point, we have $V = 2$, $E = 1$, and $R = 1$ as an edge by itself cannot divide the plane into more than one region. So $V - E + R = 2 - 1 + 1 = 2$ for now. Now if G has no more edges, we are done and we have shown $V - E + R = 2$. If G does have more edges, then choose one that is connected to one of the two existing vertices. The reason we know we can choose such an edge is that G is connected. If none of the other edges were connected to either of the two vertices we already have, then the two vertices and the edge already on the paper would form a separate component of G that is disconnected from the rest of G. OK, so we now add the second edge. If that edge connects the two existing vertices, then it divides the plane into two regions: one between the two existing edges and one outside. In this case, $V - E + R = 2 - 2 + 2 = 2$. If it only connects to one of the existing vertices then we need to add another vertex along with it. In this case, it cannot divide the plane into any new region, so $V - E + R = 3 - 2 + 1 = 2$.

If there are edges we have not added yet, then we choose another that is connected to at least one of the vertices we have already added. Just like before, if there were no such edge, the graph would have to be disconnected. Once again, there are two possibilities. If the newly added edge connects two existing vertices, then it has just enclosed a region–it has to, since the graph is connected, there was already another path between the two vertices it connects–adding 1 to R. It also adds 1 to E and does not change V, so $V - E + R$ remains unchanged. If on the other hand, the newly added edge connects an existing vertex to a new vertex, then it cannot enclose a region, because the new vertex we have just added currently stands by itself with only edge leading to it. So R is unchanged, while V and E increase by 1. Once again, $V - E + R$ remains unchanged.

And so on. As long as we have more edges to add, we can always choose the next one so it is connected to at least one of the already existing vertices because the graph is connected. Now, if the other end of the edge is also an existing vertex then this edge must cut an existing region into two regions. This adds nothing to V, 1 to E, and 1 to R. Thus $V - E + R$ does change when we add this edge. The other possibility is that the new edge connects an existing vertex to a new vertex. As we noted, such an edge cannot cut a region into two parts So we have just added 1 to V and E but did not change R. Once again, $V - E + R$ is unchanged.

Eventually we run out of edges to add. Since $V - E + R$ never changes after the first edge, it remains 2 throghout the process. This shows that $V - E + R = 2$ for any connected, planar graph, because any connected, planar graph can be built this way, one edge at a time.

You can find the book's version of this argument on pp. 402-403.

6. In class, we observed that if we flip a coin twice, the probability of having two heads is 1/4, while the probability that we do not get two consecutive heads is 3/4. Fred Flintstone and Barney Rubble play the following game of chance. They flip a coin three times and if there are at least two consective heads, Fred wins, otherwise Barney wins. So for example, Fred wins if the outcome is THH or HHH, and Barney wins it they get TTH or HTH.

   (a) (4 pts) What is the probability that Barney wins? Justify your answer.

   Each coin flip has two equally likely outcomes, so the three flips have $2 \cdot 2 \cdot 2 = 2^3 = 8$ possible (and equally likely) outcomes: HHH, HHT, THH, HTH, HTT, THT, TTH, and TTT. Of these, the first three are winning combinations for Fred, and the last five are in Barney's favor. So Barney's probability of winning is 5/8.

   (b) (4 pts) Fred and Barney play the same game, except this time they flip the coin four times. Suppose that we know that the first flip is tails. What is Barney's chance of winning the game this time? What if the first flip is heads, what is Barney's chance of winning the game then? Justify your answer.

   If the first flip is tails, then Barney will win if the next three flips do not have consecutive heads. But we already figured out in part (a) that the probability of not having consecutive heads in three flips is 5/8. So Barney's chance of winning is 5/8 if the first flip is tails.
   If the first flip is heads, then the only way Barney can win is if the next flip is tails. This has a probability of 1/2. But this is not yet enough. Then then next two flips cannot both be heads. The probability of two heads in two flips is 1/4. So the probability of the last two flips not being two heads is 3/4. Thus Barney's chance of winning is $(1/2)(3/4) = 3/8$ in this case.

   (c) (2 pts) Fred and Barney flip the coin four times again. This time, we know nothing about the outcome of the first flip. What is Barney's chance of winning? Guess what, you need to justify your answer.

   The first flip is either tails with probability 1/2 or heads, also with probability 1/2. If it is tails, then Barney has a 5/8 chance of winning, as we figured in part (b). So the probability that the first flip is tails and Barney goes on to win is $(1/2)(5/8) = 5/16$. If the first flip is heads, Barney has a 3/8 chance of winning, as we figured in part (b). So the probability that the first flip is heads and Barney goes on to win is $(1/2)(3/8) = 3/16$. The first flip is either tails or heads and these are mutually exclusive, so Barney's total chance of winning is $5/16 + 3/16 = 8/16 = 1/2$.

7. **Extra credit problem.**

   (a) (4 pts) Fred and Barney are playing the same game as in problem 6, except this time they flip the coin five times. What is the probability that Barney wins? (Hint: this is easier to calculate than you might guess if you think a bit about what you learned in parts b and c of problem 6.)

   Let us follow the same logic as in 6(c). If the first flip is tails, then Barney wins only if the next four flips do not contain consecutive heads. The probability of this is 1/2 as we noted in 6(c). On the other hand, if the first flip is heads, then Barney's only hope for winning is that the second flip is tails, which has a probability of 1/2 and then the next three flips do not contain consecutive heads, which has a probability of 5/8, as we figured in 6(a). So Barney's chance of winning a game which begins with heads is $(1/2)(5/8) = 5/16$.

Again, the first flip is either tails or heads, each with probability 1/2, so Barney's chance of winning the game is

$$\frac{1}{2}\frac{1}{2}+\frac{1}{2}\frac{5}{16}=\frac{1}{4}+\frac{5}{32}=\frac{8}{32}+\frac{5}{32}=\frac{13}{32}.$$

(b) (8 pts) Still the same game, still the same two prehistoric players. This time, Fred and Barney flip the coin n times where $n \geq 2$ is an integer. The question is once again: what is the probability that Barney wins? I will help you find the answer though. There are of course 2^n possible outcomes of the series of n coin flips. Let B_n be the number of different outcomes that do not contain consecutive heads. Show that $B_n = B_{n-1}+B_{n-2}$ by thinking about what happens if the very first flip is tails, and what happens when it is heads.

Let us use the same thinking again. If the first flip is tails, the next $n-1$ flips cannot contain consecutive heads for Barney to win. The number of outcomes of $n-1$ coin flips that do not contain consecutive heads is B_{n-1} (out of 2^{n-1} total possible outcomes). If on the other hand the first flip is heads, then the second flip must be tails and the remaining $n-2$ flips must not contain any consecutive heads, otherwise Barney loses. The number of outcomes of $n-2$ coin flips that do not contain consecutive heads is B_{n-2} (out of 2^{n-2} total possible outcomes). And of course, there is only one way for that second flip to be tails.
The first flip of the sequence is either tails or heads. In the first case, Barney has B_{n-1} winning combinations among the possible outcomes of the remaining flips. In the second case, Barney has B_{n-2} winning combinations among the possible outcomes of the remaining flips. So in total, Barney has $B_{n-1}+B_{n-2}$ winning combinations in a sequence of n coin flips. That is $B_n = B_{n-1}+B_{n-2}$.

(c) (3 pts) What are B_2 and B_3? Conclude that Barney's chance of winning the game when the coin is flipped n times is $\frac{F_{n+1}}{2^n}$, where F_{n+1} is the $n+1$-st Fibonacci number. You didn't really think the Fibonacci numbers would not show up on this exam, did you.

As problem 6 said, there are 3 winning combinations for Barney among the 4 possible outcomes if the coin is flipped twice. So $B_2 = 3$. We figured in 6(a) that Barney has 5 winning combinations among the 8 possible outcomes if the coin is flipped three times. So $B_3 = 5$. Now,

$$\begin{aligned}
B_4 &= B_3 + B_2 = 5+3 = 8\\
B_5 &= B_4 + B_3 = 8+5 = 13\\
B_6 &= B_5 + B_4 = 13+8 = 21\\
&\vdots\\
B_7 &= B_6 + B_5 = 21+13 = 34
\end{aligned}$$

Look, these are the Fibonacci numbers! Actually, they have to be. Since $B_2 = 3 = F_3$ and $B_3 = 5 = F_4$ (where we use the convention $F_1 = F_2 = 1$. so $F_2 = 2$, etc.), and the B's obey the same recursive relation as the Fibonacci numbers, the B's must be the Fibonacci numbers from this point on. Note that the indexing is a little different. In fact, $B_n = F_{n+1}$.
Now, if Fred and Barney flip the coin n times, then out of the 2^n possible outcomes B_n are in Barney's favor, so Barney's chance of winning the game is

$$\frac{B_n}{2^n}=\frac{F_{n+1}}{2^n}.$$

Go figure. That is a better story than a biologically defective one about breeding rabbits.