MCS 115 Exam 1 Solutions

1. (10 pts) Some number of pennies are spread out on a table. They lie either heads up or tails up. You are told the number of pennies that are lying heads up and are asked to arrange the pennies into two groups such that both groups have the same number of heads-up pennies. The catch is that you are blindfolded and you cannot see (or feel with your finger) which pennies are heads up. You can feel your way across the table and thus can count the total number of pennies. You are allowed to rearrange the coins, turn any of them over, and move them in any way you wish as long as the final configuration has all the pennies resting (heads or tails up) on the table.
   Following the rules above, how can you divide the pennies into two groups such that both groups have the same number of heads-up pennies? Make sure you thoroughly explain why your method always works.

Let n be the number of heads-up pennies. We know what n is. Let us split the pennies into a group of n pennies and another group that has the rest of the pennies. Call the first group A and the second group B. Some of the pennies in group A may be heads up. Let us say k of them are heads up. We do not know what k is. The other $n-k$ pernnies in group A must be tails up. We know that there are n heads-up pennies in total, and since k of them are in group A, the other $n-k$ must be in group B. So currently, group A has $n-k$ tails-up pennies and group B has $n-k$ heads-up pennies. Now we flip every penny in group A. This reverses heads-up and tails-up pennies. So the $n-k$ tails-up pennies in group A become $n-k$ heads up pennies, exactly matching the number of heads-up pennies in group B.

2. (5 pts each) You have 10 pairs of socks, 5 black and 5 blue but they are not paired up. Instead, they are all mixed up in a drawer. It's early in the morning and you don't want to turn on the lights in your dark room.

   (a) How many socks must you pull out to guarantee that you have a pair of one color?

   We need three socks. Obviously, two or fewer socks are not enough because they may all be the same color or there may not even be enough to form a pair at all. But three socks are always enough. By the Pigeonhole Principle, if you have three socks and each is one of two colors, then there must be at least two of one of the colors. (The socks are the pigeons and the colors are the pigeonholes in this case.)

   (b) How many socks must you pull out to guarantee that you have two good pairs (the socks in each pair are the same color, but the two pairs could be of different color)?

   We need five socks. Four or fewer socks are not enough because there could be three or fewer of one color and then only one or none of the other, which are not enough to form two pairs. But five socks are always enough. First, by part (a), there must be at least one matching pair among the five socks. Remove this pair. The remaining three socks must again contain a matching pair, by part (a). Now we have two good pairs.

   (c) How many socks must you pull out to be certain you have a pair of black color?

   We need 12 socks. 11 would not be enough because if we are unlucky then 10 of those are blue. And if we pull fewer than 11, they could all be blue. On the other hand, 12 are always enough because at most 10 of them are blue, so at least two of them must be black.

3. (5 pts each) In the movie The Big Lebowski, Jeff Lebowski (better known as the Dude) is asked to deliver a ransom of \$1 million in unmarked nonconsecutive \$20 bills in exchange for the kidnapped Bunny Lebowski. His friend, Walter stuffs a briefcase with his dirty underwear instead and hurls that out of the window of the car when he and the Dude are supposed to drop off the money to the kidnappers. In this problem, you are asked to judge if Walter's ruse is a realistic one.

   (a) Find a good way to estimate whether a million dollars in \$20 bills would fit inside a briefcase. It may be useful to know that a ream (500 sheets) of standard 20 lb copy paper (8.5" by 11" sheets) is about 2 inches thick.

   If we measure a \$20 bill (or any dollar bill for that matter as they are all the same size), we find that it is about 6 inches by 2.5 inches. That is 15 in^2. Dollar bills are not printed on copy paper–actually the material they are printed on is more like fabric than paper–but in terms of thickness it is not unreasonable to say that they are comparable to copy paper. So the the 50000 bills the Dude needs for the ransom would be comparable in thickness to 100 reams of copy paper, which is about 200 inches. That gives a total volume of $15 \cdot 200 = 3000$ in^3. There is a lot of flexibility about how those dollar bills can be stuffed inside a briefcase, so the pressing question is really whether the total volume could fit inside a briefcase. A briefcase that is 30" by 18" by 6" would be $30 \cdot 18 \cdot 6 = 3240$ in^3, which would be sufficient. Now, that is rather large for a briefcase, but it is not out of the question that Walter may be able to buy such a large briefcase. So yes, it may well be possible to fit a million dollars in \$20 bills would fit inside a sufficiently large briefcase.

   (b) Find a good way to estimate the weight of a million dollars in \$20 bills. Is it possible for a person to throw that weight out of a car window? It may be useful to know that a ream (500 sheets) of standard 20 lb copy paper weighs 5 lbs. (In case you are curious, it is called 20 lb paper because it is weighed in reams of 17" x 22" sheets.)

   In part (a), we said that a \$20 dollar bill has an area of about 15 in^2. So the 50000 bills the Dude needs for the ransom would have an area of 750000 in^2. We will again use copy paper as a reasonable estimate for the material dollar bills are printed on. One sheet of copy paper has an area of $8.5 \cdot 11 = 93.5$ square inches. So one ream has an area of 500 times that, which is 46750 in^2 and it weighs 5 lb. Then 750000 in^2 would weigh $(750000/46750) \cdot 5 \approx 80$ lb. Add to that the weight of the briefcase, and it would take a really strong person to hurl such a hefty briefcase in the air, but it is not entirely impossible that it could be done.

4. (5 pts each) In class, we defined the Fibonacci sequence by the following recursive formula:

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= 1 \\
F_n &= F_{n-1} + F_{n-2} \qquad \text{for } n \geq 2.
\end{aligned}
$$

   (a) In class, we noticed that the ratio F_{n+1}/F_n of two consecutive Fibonacci numbers can always be expressed as a continued fraction

$$
\frac{F_{n+1}}{F_n} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots \, \cfrac{}{1 + \frac{1}{1}}}}}
$$

   for every integer $n \geq 1$. Explain how we know this.

Notice that

$$\frac{F_1}{F_0} = \frac{1}{1}$$

$$\frac{F_2}{F_1} = \frac{2}{1} = \frac{1+1}{1} = \frac{1}{1} + \frac{1}{1} = 1 + \frac{1}{1}$$

$$\frac{F_3}{F_2} = \frac{3}{2} = \frac{2+1}{2} = \frac{2}{2} + \frac{1}{2} = \frac{2}{2} + \frac{1}{\frac{2}{1}} = 1 + \frac{1}{1+\frac{1}{1}}$$

$$\frac{F_4}{F_3} = \frac{5}{3} = \frac{3+2}{3} = \frac{3}{3} + \frac{2}{3} = \frac{3}{3} + \frac{1}{\frac{3}{2}} = \frac{1}{1+\frac{1}{1+\frac{1}{1}}}$$

$$\vdots$$

$$\frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = \frac{F_n}{F_n} + \frac{F_{n-1}}{F_n} = 1 + \frac{1}{\frac{F_n}{F_{n-1}}} = 1 + \frac{1}{\boxed{1 + \frac{1}{1 + \frac{1}{1 + \ddots_{1+\frac{1}{1}}}}}}$$

and so on. The pattern continues forever because we keep substituting the previous ratio into the next: the framed part in $\frac{F_{n+1}}{F_n}$ is F_n/F_{n-1}.

Notice that the argument above is an informal version of mathematical induction. We talked about mathematical induction when we proved that every natural number is interesting.

(b) We also noticed that F_{n+1}/F_n appears to be getting closer to some number near 1.6 as n gets large. Assuming that it is indeed true that there is a number ϕ that F_{n+1}/F_n approaches as n gets large, use the result in part (a) to show that ϕ must satisfy the equation

$$\phi = 1 + \frac{1}{\phi}.$$

If n is large, then

$$\frac{F_{n+1}}{F_n} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ddots_{1+\frac{1}{1}}}}} \approx \phi.$$

But if n is large, then $n-1$ is also large. So it is also true that

$$\frac{F_n}{F_n - 1} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ddots_{1+\frac{1}{1}}}}} \approx \phi.$$

Now because the part in the framed box F_n/F_{n-1}, which is also approximately ϕ. The two approximations above can be made to be as precise as we want by making n larger and larger. But ϕ is a fixed number and its value does not depend on n. So the only way the value of ϕ can be as close as we want to the value of $1 + 1\phi$ is if they are equal. Therefore

$$\phi = 1 + \frac{1}{\phi}.$$

(c) Solve the equation in part (b) to find the exact value of ϕ. Note that exact value means it is not a decimal approximation.

Multiply both sides of the equation $\phi = 1+1/\phi$. We know that ϕ cannot be 0, otherwise we could not have it in the denominator in $1/\phi$. So we get

$$\phi^2 = \phi + 1.$$

This is a quadratic equation for ϕ and we can solve it using the quadratic formula. But first, we need to move all of the terms to one side of the equation:

$$\phi^2 - \phi - 1 = 0.$$

By the quadratic formula

$$\phi = \frac{1 \pm \sqrt{(-1)^2 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Since $\sqrt{5} > 1$, one of these values is negative. In fact, we know we are looking for a number near 1.6, so the correct value must be

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618.$$

Hence the exact value of ϕ is $\frac{1+\sqrt{5}}{2}$, which is also known as the golden ratio.

5. **Extra credit problem.** In class we solved the Meanie Genie problems with 12 stones and briefly noted that the problem could be solved similarly if there were 13 stones. Now suppose that Murray, the genie in fact gives you 13 stones, of which one has a precious gem hidden inside it. The stones look identical except that the stone with the hidden gem is slightly different in weight from the other twelve stones, which all weigh the same. The weight difference is too small to feel by hand and the only way to detect it is to use a percise balance scale. Murray gives you three simple balance scales, but each will work only once.

(a) (5 pts) Describe how you can find the gemstone with the help of the scales and explain why your method does indeed find the gemstone.

For easier reference, number the stones 1 through 13. We will start by putting stones #1, #2, #3, and #4 on one side of a scale and stones #5, #6, #7, and #8 on the other side. There are two cases: the scale is balanced or the scale tips. The latter can be solved exactly the way we did in class. The former can also be solved similarly to how we solved the 12-stone problem in class, but now we have one more stone to deal with.

The first scale is balanced: In this case, the gemstone is obviously not on the scale, so it must be among stones #9-13. Now, we put stones #9, #10, and #11 on one side of the second scale and weigh them against stones #1, #2, and #3. There are two cases:

The second scale is balanced: Then the gem is not on the scale, so it must be in stone #12 or #13. We can now use the third scale to weigh stone #12 against #1. If the scale tips, then the gem is in stone #12. If not, then it must be #13.

The second scale tips: In this case, we know the gemstone must be on the scale and we actually know it is among stones #9, #10, and #11. Which way the scale tips tells us if the stone with the gem is heavier or lighter than the rest. Suppose it is heavier. We use the third scale to weigh stones #9 and #10 against each other. If one is heavier than the other, then that must

have the gemstone. If they are balanced, the gem must be in stone #11 and we already know it is heavy.
If the second scale indicates that the gemstone is lighter than the rest, it can be found among stones #9, #10, and #11 similarly.

The first scale tips: In this case, the gemstone must be among stones #1-8. Suppose the scale tips in the direction of stones #5, #6, #7, and #8. (If it tips the other way, you can always relabel the stones so that the heavier side of the scale has stones #5, #6, #7, and #8.) We now put stones #1, #2, #3, and #5 on one side of the second scale and weigh them against stones #4, #9, #10, and #11. There are three possibilities:

The second scale is balanced: Then the gemstone must be among stones #6, #7, and #8. We also know that it is heavy, since these stones were on the heavy side of the first scale. We use the third scale to weigh stones #6 and #7 against each other and can identify the gemstone just like we did when we were looking for it among stones #9, #10, and #11.

The second scale tips toward stones #4, #9, #10, and #11: In this case, the gemstone must be on the scale. We know it is not stone #9, #10, or #11. It cannot be stone #4 either because stone #4 was on the light side of the scale in the first measurement and now it is on the heavy side. It cannot be stone #5 either for a similar reason. So the gemstone is among stones #1, #2, and #3. We can also tell that the gemstone is light. We use the third scale to weigh stones #1 and #2 against each other and can identify the gemstone just like we did when we were looking for it among stones #9, #10, and #11.

The second scale tips toward stones #1, #2, #3, and #5: This must be either because the gem is in stone #4 and is light or because the gem is in stone #5 and is heavy. We now use the third scale to weight stone #4 against #1. If stone #4 is lighter, then it is the one with the gemstone. Otherwise the gem is in stone #5 and we know it is heavier than the rest.

(b) (10 pts) Is it possible both to find the gemstone and figure out if it is heavier or lighter than the rest of the stones? Note that the question is about the possibility of doing this. We are not asking whether the solution you give to part (a) can detect if the gemstone is heavier or lighter, but whether it is possible–with your method or some other clever method–to find the gemstone and also deduce if it is heavier or lighter than the rest of the stones.

Looking carefully at the method in part (a), we do indeed find out if the gemstone is heavier or lighter than the rest in almost all cases, except if it turns out to be #13. Could there be a better approach which tells us in all cases? We will argue that it cannot be done.
In the first measurement, we would place some of the stones on one side of the first scale and the same number of stones on the other side. This is the only move that makes sense to do. If we put different numbers of stones on the two sides of the scale, the scale would likely just tip toward the side with the more stones, but this would tell us nothing meaningful. So we will have to weigh and even number of stones on the first move. If the scale tips, we know that the gemstone is among the stones on the scale. Now, we have two more measurements to find it. As we argued in class, two measurements have nine distinguishable outcomes as each scale could tip to the left or to the right or stay balanced, and for each outcome of the second measurement, the third measurement has

three possible outcomes. This means that the last two measurements can distinguish at most among nine stones. If we put any more than nine stones on the first scale, and the scale tips, we may not be able to find the gemstone with the remaining two measurements. But nine is an odd number, so really, we can put at most eight stones on the first scale.

Now, if the first measurement is balanced, then the gemstone is among the stones not on the scale. Since we placed at most eight stones on the first scale, there must be at least five stones not on the scale. If we now want to find the gemstone and also figure out if it is heavier or lighter than the rest, we need to be able to distinguish among at least ten different possibilities because the gemstone could be any one of the stones not on the scale, and in each possible case, it could be either heavy or light. But as we already argued, the remaining two measurements cannot possibly distinguish among more than nine possibilities. So we will not be able to both identify the gemstone and learn whether it is heavier or lighter than the other stones.