MCS 115 Exam 2 Solutions

Apr 26, 2018

1. (10 pts) Let S stand for the set of all natural numbers that are perfect squares, so $S = \{1, 4, 9, 16, 25, 36, 49, 64, \ldots\}$. Show that the set S and the set of all natural numbers $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ have the same cardinality by describing an explicit one-to-one correspondence between the two sets. Make sure you justify your answer by explaining why your correspondence is one-to-one, i.e. how we know that each natural number n corresponds to some element in S and each element s of S corresponds to some element of $\mathbb{N}$.

   Hint: When we say "describing an explicit one-to-one correspondence," we mean a description that allows us to tell which element of S corresponds to any element n of $\mathbb{N}$. You can do this in words, or with a formula, but you need to do more than just specify a few pairs and leave it to the reader to guess what the correspondence actually is.

   We can pair up

   $$1 \leftrightarrow 1, 2 \leftrightarrow 4, 3 \leftrightarrow 9, 4 \leftrightarrow 16, \ldots, n \leftrightarrow n^2, \ldots$$

   To show that $n \leftrightarrow n^2$ is a one-to-one correspondence, we need to show that every natural number n corresponds to a perfect square and vice versa. If n is any natural number, then n^2 is a perfect square, so $n^2 \in S$. Hence n does indeed correspond to an element in S. Now, if s is a perfect square, i.e. s is the square of some integer n. We can always choose n to be a non-negative integer. In fact, since $0 \notin S$, we can always choose n to be a positive integer. Thus any element of s corresponds to an element of $\mathbb{N}$. Therefore $n \leftrightarrow n^2$ is a one-to-one correspondence between $\mathbb{N}$ and S, so these two sets have the same cardinality.

2. Consider the following infinite collection of real numbers:

   $$\begin{gathered} 0.12345678910111213141516171 8\ldots \\ 0.2468101214161820222426283032\ldots \\ 0.369121518212427303336394245\ldots \\ 0.4812162024283236404448525660\ldots \\ 0.510152025303540455055606570\ldots \\ \vdots \end{gathered}$$

   (a) (5 pts) Describe in your own words how these numbers are constructed (that is, describe the procedure for generating this list of numbers).

   The number in the n-th row is constructed by listing the multiples of n as $n, 2n, 3n, \ldots$ after the decimal point and using these as the digits of the number.

   (b) (10 pts) Using Cantor's diagonalization argument, find a number not on the list. Justify your answer, that is explain how we know that the number you found cannot be on the list.

   We can do the same thing we did in class: we will construct a number m by going down the diagonal

$$
\begin{aligned}
&0.\boxed{1}2345678910111213141516171 8\ldots\\
&0.2\boxed{4}68101214161820222426283032\ldots\\
&0.36\boxed{9}12151821242730333639424 5\ldots\\
&0.481\boxed{2}16202428323640444852566 0\ldots\\
&0.5101\boxed{5}2025303540455055606570\ldots\\
&\vdots
\end{aligned}
$$

and for every digit that is 2 we write down a 4 and for every digit that is not 2 we write down a 2:

$$m = 0.22242\ldots$$

We know this number cannot be on the list because it differs from every number on the list in at least one digit. Specifically, suppose m were on the list, say in the n-th row. In the process of constructing m above, we would have looked at the n-th digit of the n-th number. If this number were m, then that digit would have to be 2 or 4. But if it were 2, we actually would have used 4 as the n-th digit of m. And if the digit were 4, then we would have used 2 instead. So no matter what, the n-th number on the list cannot be the same as m.

3. (a) (4 pts) Let n be a positive integer and let x and y be integers. Explain what $x \equiv y \pmod{n}$ means.

   $x \equiv y \pmod{n}$ means that $x - y$ is a multiple of n. Another way to say this is that $x - y$ is divisible by n. Yet another way to say this is that x and y have the same remainders when divided by n.

   (b) (6 pts) We noted that division modulo n may not be well-defined, i.e. even if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, it could well be that $a/c \not\equiv b/d \pmod{n}$. Give an example of this: choose a modulus n and find specific integers a, b, c, d such that $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, but even though a/c and b/d are integers, they are not equivalent modulo n.

   There are many examples. Here is one: $n = 6$, $a = 10$, $b = 16$, $c = 2$, $d = 8$. Then

$$
\begin{aligned}
10 &\equiv 16 \pmod{6} && \text{because } 16 - 10 = 6\\
2 &\equiv 8 \pmod{6} && \text{because } 8 - 2 = 6\\
\frac{10}{2} = 5 &\not\equiv 2 = \frac{16}{8} \pmod{6} && \text{because } 5 - 2 = 3
\end{aligned}
$$

4. In class, we defined a golden rectangle as a rectangle with sides a and b such that

$$\frac{a+b}{b} = \frac{b}{a}.$$

   (a) (7 pts) Use the above definition to find the exact value of b/a.

   First, let $\phi = b/a$. Now notice that

$$\frac{a+b}{b} = \frac{a}{b} + \frac{b}{b} = \frac{1}{\frac{b}{a}} + 1 = \frac{1}{\phi} + 1.$$

So

$$
\begin{aligned}
\frac{a+b}{b} &= \frac{b}{a} \\
\frac{1}{\phi} + 1 &= \phi && \text{multiply both sides by } \phi \\
\phi + 1 &= \phi^2 && \text{subtract } \phi + 1 \text{ from both sides} \\
0 &= \phi^2 - \phi - 1
\end{aligned}
$$

By the quadratic formula

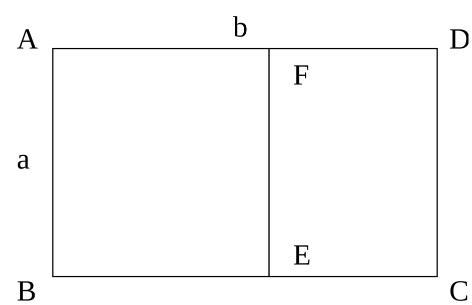
$$\phi = \frac{1 \pm \sqrt{(-1)^2 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Since $\sqrt{5} > 1$, we can tell $\frac{1-\sqrt{5}}{2}$ is negative. But b and a were the sides of a rectangle, so they must be positive, and hence b/a must be positive too. So the correct value is

$$\frac{b}{a} = \frac{1+\sqrt{5}}{2}.$$

If you are paying attention, you may have noticed this is almost exactly the same argument you were asked to give on the last exam, only F_{n+1} and F_n are replaced by b and a.

(b) (8 pts) Let the rectangle $ABCD$ in the diagram below be a golden rectangle. Suppose $ABEF$ is a square. Prove that rectangle $CDFE$ is also a golden rectangle.



We gave three different arguments in class. For the sake of completeness, here they are all. The first two are almost the same.

First, note that $EB = b - a$. So we need to show $\frac{a}{b-a} = \phi$. We will instead prove $\frac{b-a}{a} = \frac{1}{\phi}$ and then taking reciprocals gives $\frac{a}{b-a} = \phi$.

$$
\begin{aligned}
\frac{b-a}{a} &= \frac{b}{a} - \frac{a}{a} \\
&= \phi - 1 = \frac{1+\sqrt{5}}{2} - 1 = \frac{1+\sqrt{5}-2}{2} = \frac{\sqrt{5}-1}{2} \\
&= \frac{\sqrt{5}-1}{2}\,\frac{\sqrt{5}+1}{\sqrt{5}+1} = \frac{(\sqrt{5}-1)(\sqrt{5}+1)}{2(\sqrt{5}+1)} = \frac{\sqrt{5}^2 - 1^2}{2(\sqrt{5}+1)} \\
&= \frac{5-1}{2(\sqrt{5}+1)} = \frac{4}{2(\sqrt{5}+1)} = \frac{2}{\sqrt{5}+1} = \frac{1}{\phi}
\end{aligned}
$$

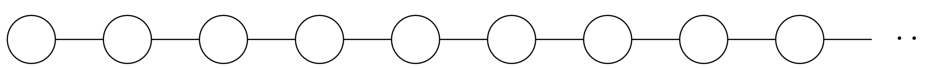
Second,

$$
\begin{aligned}
\frac{a}{b-a} &= \frac{\frac{a}{a}}{\frac{b-a}{a}} = \frac{1}{\frac{b}{a}-\frac{a}{a}} = \frac{1}{\phi-1} = \frac{1}{\frac{1+\sqrt{5}}{2}-1} = \frac{1}{\frac{1+\sqrt{5}-2}{2}} \\
&= \frac{2}{\sqrt{5}-1} = \frac{2}{\sqrt{5}-1}\,\frac{\sqrt{5}+1}{\sqrt{5}+1} = \frac{2(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)} 2(\sqrt{5}+1) \\
&= \frac{2(\sqrt{5}+1)}{\sqrt{5}^2-1^2} = \frac{2(\sqrt{5}+1)}{5-1} = \frac{2(\sqrt{5}+1)}{4} = \frac{\sqrt{5}+1}{2} = \phi
\end{aligned}
$$

Third, we start with $\frac{a+b}{b} = \frac{b}{a}$:

$$
\begin{aligned}
\frac{a+b}{b} &= \frac{b}{a} && \text{multiply both sides by } ab \\
\frac{a+b}{\cancel{b}}\,a\cancel{b} &= \frac{b}{\cancel{a}}\,\cancel{a}b \\
(a+b)a &= b^2 \\
a^2 + ba &= b^2 && \text{subtract } ba \text{ from both sides} \\
a^2 &= b^2 - ba \\
a^2 &= b(b-a) && \text{divide both sides by } b-a \\
\frac{a^2}{b-a} &= b && \text{divide both sides by } a \\
\frac{a}{b-a} &= \frac{b}{a} \\
\frac{a}{b-a} &= \phi
\end{aligned}
$$

5. **Extra credit problem.** On your homework, you considered infinite chains of circles like the one below:



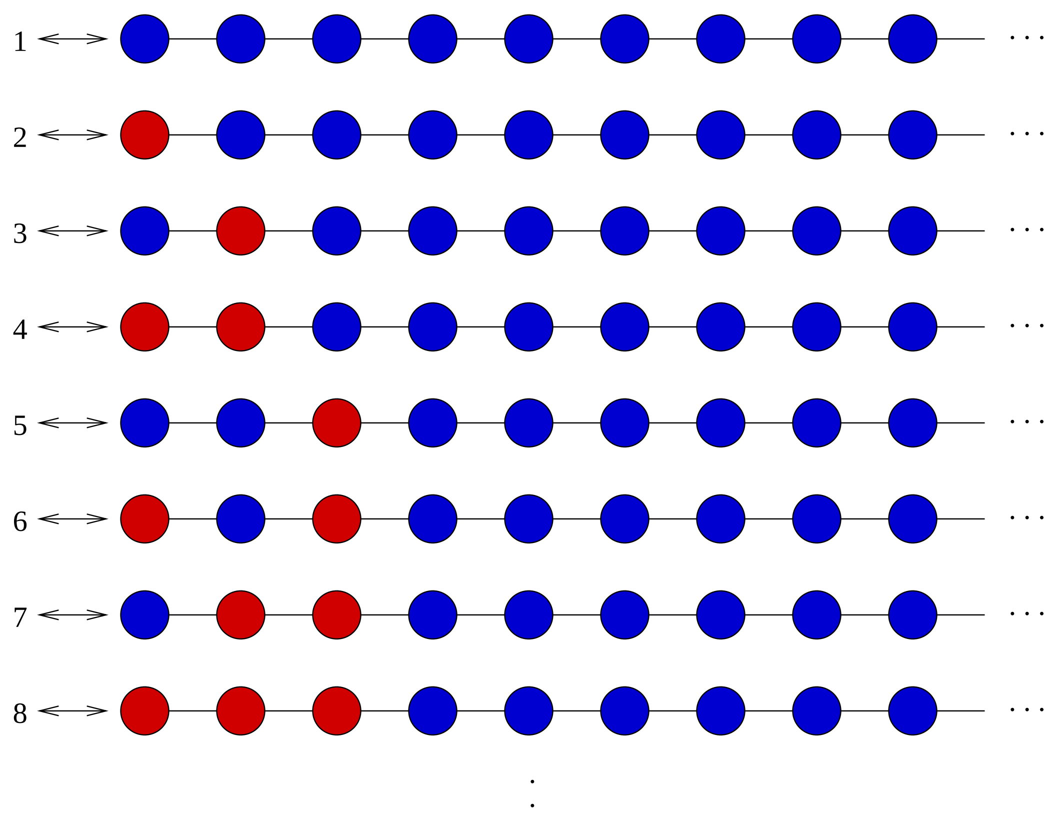
and showed that the number of ways to color such a chain by making each circle either red or blue has greater cardinality than the set of natural numbers $\mathbb{Z}^+$. Now suppose that you are again to color the chain by making each circle red or blue, but you can only color finitely many of the circles red. So for example, the coloring in which the circles alternate between red and blue is not allowed because it would require coloring infinitely many circles red.

(a) (5 pts) Show that there are still infinitely many different ways to color a chain.

Notice that one such coloring is that we color the first circle red and all the circles after that blue. Another, different coloring is that we color the first two circles red and all the circles after those blue. A third one that the first three circles are red and all the circles after those are blue. And so on. Let C_n for $n = 1, 2, 3, \ldots$ be the chain in which the first n circles are red and the remaining circles are blue. Such a chain has finitely many red circles of course. But there are infinitely many such chains, one for each positive integer $n = 1, 2, 3, \ldots$. Those are not the only ways to color a chain, but they are already infinitely many ways to color a chain.

(b) (10 pts) Does the number of ways to color a chain have the same cardinality as the set of natural numbers? If you think it does, show that it does by finding an appropriate one-to-one correspondence between all such colorings and the natural numbers. If you do not think it does, find an argument to show why no such one-to-one correspondence can exist.

We will show that the number of ways to color a chain has the same cardinality as the set of natural numbers. To construct a one-to-one correspondence between the natural numbers and the colorings of the chain, we will list all of the possible colorings in an infinite list. As we pointed out in class, such an infinite list is a one-to-one correspondence with the natural numbers because it assigns to each coloring its position in the list, and that assignment is one-to-one.
First, notice that since each coloring of the chain is allowed to use only finitely many red circles, there is always a point along the chain after which all of the circles are blue. So we first list the chain whose circles are all blue. Then we list the chains in which the last red circle is the first circle and all of the circles after that are blue. There is actually only one such chain. Then we list the chains in which the last red circle is the second one. These are the next two chains on the list. Then we list the ones in which the last red circle is the third one. There are four such chains, the next four on the list below. And so on. Here is what the beginning of our list looks like:



Every chain with only finitely many red circles has a last red circle, so it will be listed eventually. How far down the list it appears determines which natural number corresponds to it. This way, each coloring corresponds to a natural number. Since there are infinitely many colorings, each natural number also corresponds to some coloring.

Notice that the way we organized the chains into an infinite list is similar to how one would list the rational numbers between 0 and 1 to show that they are equinumerous with the natural numbers, which is what Mindscape 3.2.26 asked you to do on your homework.