MCS 118 Exam 1 Solutions

1. (5 pts each) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions.
   (a) If f and g are both even functions, is the product fg even?

   We know $f(-x) = f(x)$ and $g(-x) = g(x)$ for all $x \in \mathbb{R}$. So
   $$fg(-x) = f(-x)g(-x) = f(x)g(x) = fg(x)$$
   for all $x \in \mathbb{R}$. This shows fg is an even function.

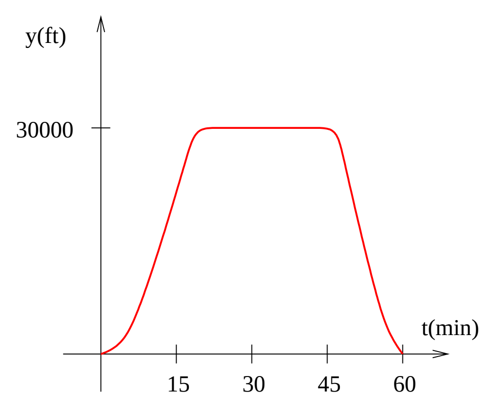
   (b) Is fg odd if f is even and g is odd?

   We know $f(-x) = f(x)$ and $g(-x) = -g(x)$ for all $x \in \mathbb{R}$. So
   $$fg(-x) = f(-x)g(-x) = f(x)\big(-g(x)\big) = -f(x)g(x) = -fg(x)$$
   for all $x \in \mathbb{R}$. This shows fg is an odd function.

2. (5 pts each) An airplane takes off from an airport and lands an hour later at another airport, 400 miles away. If t represents the time in minutes since the plane has left the terminal, let $y(t)$ be the altitude of the plane at time t.
   (a) Sketch a possible graph of $y(t)$. Remember to label your axes with a proper scale and units. Justify your answer by explaining the shape of your graph.

   The first question is how we would measure altitude. Is it the vertical distance between the airplane and the ground it is currently flying over, or altitude over a standard reference altitude, such as sea level? The problem with the first one is that altitude would fluctuate as the plane flies over hills and mountains even though the plane does not really go up or down relative to its flight plath. For this reason, altitude over sea level is a more practical choice. So that is what we will do. So the plane starts out at the altitude above sea level of the airport where it is taking off. For the sake of simplicity, let us assume that both the departure and the arrival airports are at about 0 ft, although there are of course airports at much higher elevations too, and even some below sea level.

   The plane spends the first few minutes taxiing to the runway, then accelerating down the runway until it lifts off. During this time its altitude remains constant. Once the plane lifts off, its altitude increases. It is reasonable to assume that it does so at an approximately steady rate while the plane climbs. Once it reaches its cruising altitude, it levels off and stays at that altitude for a while. Eventually the plane descends, during which time its altitude decreases. It is reasonable to think that it does so at a steady rate, until the plane gets close to the runway. Then the plane spends some time taxiing to the arrival gate, during which time its altitude does not change.

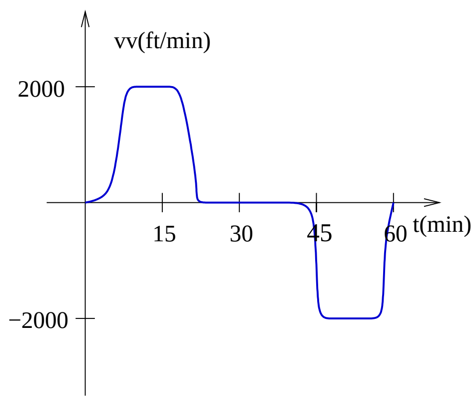


   At the last phase of the landing, the pilot will slow down the rate of decrease of the altitude so the plane can gently touch down on the runway, instead of crashing into the pavement.

   We need to decide what is a reasonable cruising altitude and how long it would take he plane to climb there and then to descend from it. First, note that an airplane that can cover a distance of 400 miles in a hour is not grandpa's barnstorming biplane. It is a high performance plane, probably some kind of a commercial or military plane. Let us say it is a commercial plane. If you have ever flown in a commercial plane and listened to the pilot's announcements, you probably know that a typical cruising altitude for a commercial aircraft is somewhere between 25000 ft and 35000 ft. I will choose 30000 ft. And it takes about 15 mins to get up there and 15 mins to descend and arrive at the gate.

(b) Sketch a possible graph of the vertical velocity as a function of t. Remember to label your axes with a proper scale and units. Justify your answer by explaining the shape of your graph.

The vertical velocity is 0 while on the ground and also while the plane cruises at a constant altitude. It is positive during climb and negative during descent and landing. As we said in part (a), it is reasonable to assume that altitude increases/decreases at a steady rate during climb and descent, and hence the vertical velocity is constant during these phases of the flight. Since we measured altitude in ft, a reasonable unit for vertical velocity is ft/min. So what is a reasonable value during climb? We said that it takes about 15 mins to reach the cruising altitude of 30000 ft, so vertical velocity may be 2000 ft/min during climb, and similarly, -2000 ft/min during descent.



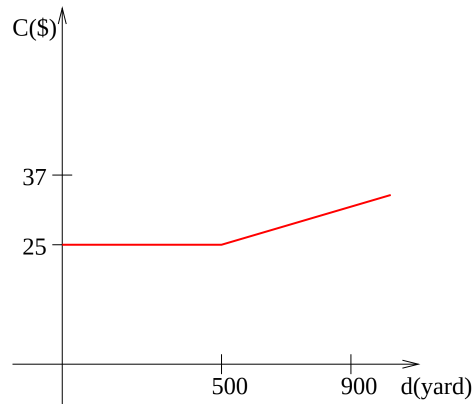
3. (5 pts each) You rent a unicycle from Krusty the Clown's Unicycle Rental. The rental costs \$25 for the day and includes the first 500 yards of riding. There is a charge for every additional yard beyond that. Yes, Krusty's unicycles have odometers. You know from Sideshow Bob that when he rented a unicycle from Krusty and rode it 900 yards, he paid \$37.

(a) Write down an expression for the rental charge C as a function of the distance d that the unicycle is ridden.

If $d \leq 25$, then $C = 25$. Sideshow Bob paid an additional \$12 for the additional 400 yards, so the charge per yard should be \$0.03. Hence if $d > 500$, then another $0.03(d - 500)$ is charged. So

$$C(d) = \begin{cases} 25 & \text{if } 0 \leq d \leq 500 \\ 25 + 0.03(d - 500) & \text{if } d > 500. \end{cases}$$

(b) Graph $C(d)$. Remember to label your axes.



4. (5 pts each)
   (a) The domain of the function $f(x) = \sqrt{c - x^2}$ is $(-5, 5)$. What does this tell you about the value of c?

   The function must have a value at every $x \in (-5, 5)$. So $c - x^2$ must be nonnegative whenever $-5 < x < 5$. When $-5 < x < 5$, then $0 \leq x^2 < 25$. Although x^2 is never 25, it gets very close to 25. In fact, it reaches any number that is less than 25. Therefore c cannot be less than 25, that is $c \geq 25$.

   (b) Give an example of a rational function whose domain is $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$.

   A rational function is of the form $f(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials. Since the domain includes all real numbers except -3 and 2, $q(x)$ must be nonzero for any real number $x \neq -3, 2$. So $q(x)$ could be $q(x) = (x + 3)(x - 2)$, as this q is 0 at exactly $x = -3$ and $x = 2$. Since any polynomial has a value at any real number x, p can be any polynomial, say $p(x) = 2x^2 + 5$. So

$$f : (-\infty, -3) \cup (-3, 2) \cup (2, \infty) \to \mathbb{R}$$
$$f(x) = \frac{2x^2 + 5}{(x + 3)(x - 2)}$$

   would be a good example of such a rational function.
   Note that it is not required that $q(x) = 0$ at $x = -3$ and $x = 2$. So we could have also chosen a polynomial fpr q that has no zeroes at those points, as long as it is not 0 anywhere else, e.g. $q(x) = x^2 + 1$,

5. (5 pts each) **Extra credit problem.** Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions.
   (a) If f and g are both increasing functions, is the product fg increasing?

   The product fg need not be increasing. For example, let $f(x) = x$ and $g(x) = x$. Then f and g are both increasing functions since for any $x_1 < x_2$, it is true that $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$. But $fg(x) = f(x)g(x) = x^2$ is not an increasing function, as $-3 < -2$ but $9 = fg(-3) > f(-2) = 4$.

   (b) Can f be both increasing and an even function?

   No, it cannot. If f is an even function, then its graph is symmetric about the y-axis. Because of this symmetry, if the graph is increasing on the positive side of the x-axis, then it must be decreasing on the negative side. Alternately, note that if f is an even function, then $f(-x) = f(x)$. In particular, $f(-1) = f(1)$. If f is an increasing function, then $f(-1) < f(1)$ since $-1 < 1$. But f cannot satisfy both of these at the same time.