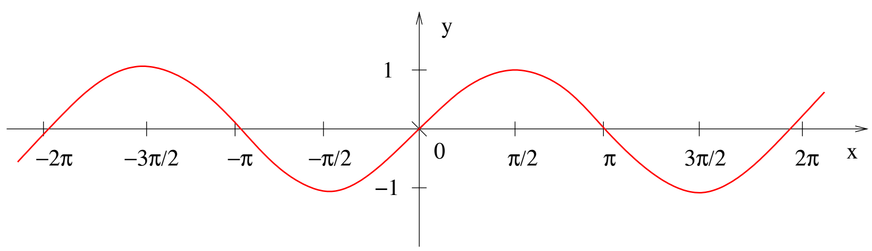
MCS 118 Exam 2 Solutions

Nov 9, 2018

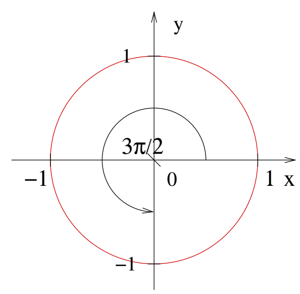
1. (10 pts) Let $f(x) = \frac{x}{1+x}$ and $g(x) = \sin(2x)$. Find $f \circ g(x)$ and the largest possible subset of the real numbers that could be the domain of $f \circ g$.

$$f \circ g(x) = f(g(x)) = f(\sin(2x)) = \frac{\sin(2x)}{1 + \sin(2x)}.$$

To find the domain, note that $\sin(2x)$ exists for any real number x because the domain of the sine function is $\mathbb{R}$. But the division can only be done if $1 + \sin(2x) \neq 0$. For what values of x is $1 + \sin(2x) = 0$? This happens whenever $\sin(2x) = -1$. Looking at the graph of the sine function



or the unit circle



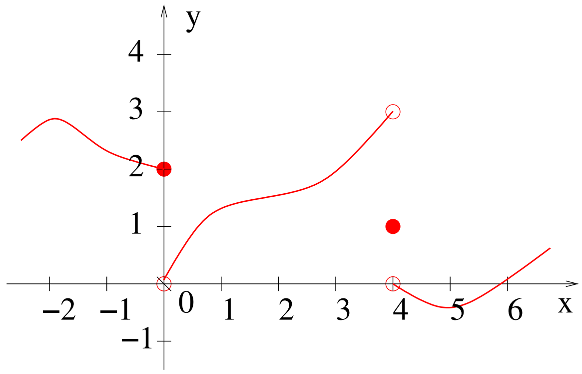
shows that $\sin(2x) = -1$ whenever $2x = \frac{3\pi}{2} + k2\pi$ where $k \in \mathbb{Z}$. Divide by 2 to get $x = \frac{3\pi}{4} + k\pi$. These are the values of x that makes the denominator of $f \circ g(x)$ equal 0, so they must be excluded from the domain. Other real numbers do not cause any trouble. So

$$D(f \circ g) = \{x \in \mathbb{R} \mid x = \frac{3\pi}{4} + k\pi \text{ where } k \text{ is any integer}\}.$$

2. (10 pts) Sketch the graph of an example of a function f that satisfies all of the conditions

$$\lim_{x \to 0^-} f(x) = 2, \qquad \lim_{x \to 0^+} f(x) = 0, \qquad f(0) = 2,$$
$$\lim_{x \to 4^-} f(x) = 3, \qquad \lim_{x \to 4^+} f(x) = 0, \qquad f(4) = 1.$$

Be sure to explain how you came up with the graph.



Because $f(0) = 2$ and $f(4) = 1$, the point $(0,2)$ and $(4,1)$ are on the graph. Since $\lim_{x\to 0^-} f(x) = 2$ the value of f must approach 2 as x approaches 0 from the left. Similarly, $\lim_{x\to 0^+} f(x) = 0$ says that the value of f approaches 0 as x approaches 0 from the right, but $f(0) = 2$, so the point $(0,0)$ is not part of the graph, which we indicated with an empty circle there. Because $\lim_{x\to 4^-} f(x) = 3$ and $\lim_{x\to 4^+} f(x) = 0$, the graph must also approach the point $(4,3)$ from the left and $(4,0)$ from the right. But neither of these points belong to the graph, which is indicated by empty circles there.

3. (5 pts each) Suppose f is an even function.
   (a) If g is any other function and $h = f \circ g$, is h always an even function?

   No, h is not always even. For example, let $f(x) = x^2$ and $g(x) = x + 1$. Then $h(x) = f \circ g(x) = (x+1)^2$. Notice that $h(1) = 4$ and $h(-1) = 0$, so $h(-1) \neq h(1)$. Therefore $h(-x) = h(x)$ cannot be true for every x. Therefore h is not an even function.

   Note that one specific example is enough evidence to show that h is not always an even function.

   (b) If g is any other function and $h = g \circ f$, is h always an even function?

   In this case, h is even. Since $f(-x) = f(x)$ for every x,
$$h(-x) = g(f(-x)) = g(f(x)) = h(x)$$
   for every x. Hence h is an even function.

4. (5 pts each)
   (a) Let f be a function and a and L real numbers. State the formal $(\delta - \epsilon)$ definition of
$$\lim_{x\to a} f(x) = L.$$

   Let f be a function and let a be a number such that some neighborhood of a except possibly a itself is contained in the domain of f. If for every $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$ then
$$\lim_{x\to a} f(x) = L.$$

   (b) Give an example of a function f and a number a such that some neighborhood of a (that is an open interval surrounding a) except possibly a itself lies within the domain of f, and $\lim_{x\to a} f(x)$ does not exist. Of course, you will have to explain why the limit does not exist. Note that you will have to specify the domain of f so that you can check that some open interval surrounding a is in fact contained in it.

One example we gave in class was $f(x) = \sin(\pi/x)$ as $x \to 0$. Indeed, any number other than 0 is in the domain of f, as any number $x \neq 0$ results in some number π/x and the sine function can take any real number as its input. So any open interval that surrounds 0 lies in the domain. But

$$\lim_{x\to 0} \sin(\pi/x)$$

does not exist because $\sin(\pi/x)$ keeps running through values between -1 and 1 as x gets close to 0 and never approaches one particular number.

5. (10 pts) **Extra credit problem.** Let $f : \mathbb{R}^* \to \mathbb{R}$ be any odd function (remember $\mathbb{R}^*$ is the set of nonzero real numbers). Prove that either $\lim_{x\to 0} f(x)$ does not exist or $\lim_{x\to 0} f(x) = 0$. (Hint: look at the left and right limits of f at 0.)

If $\lim_{x\to 0} f(x)$ does not exist, then there is nothing to prove. So suppose that $\lim_{x\to 0} f(x)$ exists. Let $L = \lim_{x\to 0} f(x)$. Then the one-sided limits $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$ must also exist and must be equal to L. Suppose L is positive. Then $f(x)$ must get arbitrarily close to L for every x that is sufficiently close to 0. Since L is positive, when $f(x)$ is close to L, it must be positive too. But x is exactly the same distance from 0 as $-x$ is, so if x is sufficiently close to 0, so is $-x$. Therefore $f(-x)$ must be close to L as well, and therefore $f(-x)$ must also be positive. But then $f(-x) = -f(x)$ cannot be true and f could not be an odd function. Therefore L cannot be positive. An analogous argument shows that L cannot be negative either. Therefore $L = 0$. So if $\lim_{x\to 0} f(x)$ exists, it must be 0.

Just in case you are curious if both of these can indeed happen, they can. First, let $f(x) = 1/x$. It is indeed an odd function, as

$$f(-x) = \frac{1}{-x} = -\frac{1}{x} = -f(x).$$

The limit $\lim_{x\to 0} 1/x$ does not exist because as x gets close to 0, the values of $1/x$ become really large if $x > 0$ or very negative if $x < 0$, and they do not approach a particular number.

Now, let $f(x) = x^3$. This is also an odd function, as we saw in class. And

$$\lim_{x\to 0} x^3 = 0$$

because x^3 gets closer and closer to 0 as $x \to 0$.