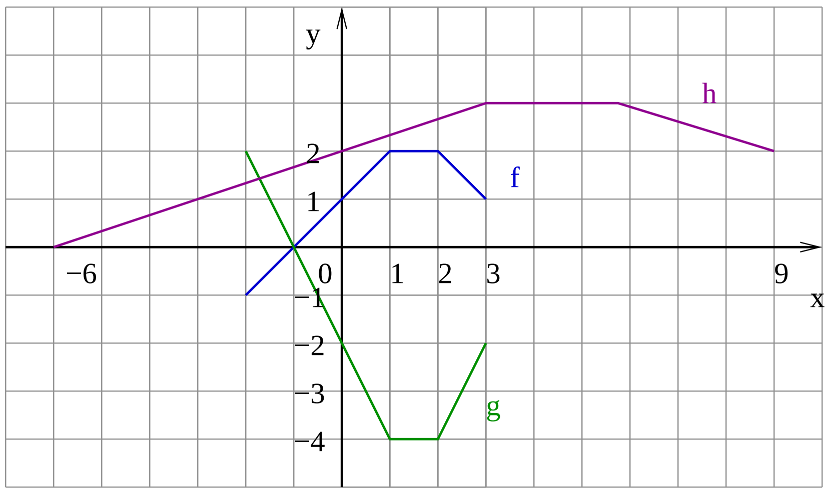
MCS 118 Final Exam Solutions

1. (5 pts each) The graph of f is given. Draw the graphs of the following functions. Feel free to use the same coordinate axes already given in the diagram, but be sure to explain what you did to the graph of f to construct the new graph.
   (a) $g(x) = -2f(x)$
   (b) $h(x) = f\left(\frac{1}{3}x\right) + 1$



To get the graph of $g(x)$, stretch the graph of $f(x)$ by a factor of 2 in the vertical direction and reflect it across the horizontal axis. To get the graph of $h(x)$, stretch the graph of $f(x)$ by a factor of 3 in the horizontal direction and shift it up by a unit.

2. (10 pts) Prove the statement using the $\delta - \epsilon$ definition of a limit:

$$\lim_{x \to 0} x^3 = 0.$$

We need to show that for every $\epsilon > 0$, there is a $\delta > 0$ such that if $0 < |x - 0| < \delta$ then $|x^3 - 0| < \epsilon$. We start by figuring out what a good choice for δ is depending on ϵ. We want $|x^3 - 0| < \epsilon$, which is the same thing as

$$|x^3| < \epsilon.$$

Use the fact that $|ab| = |a||b|$ for any real numbers a, b to write $|x^3| = |x|^3$. So we really want

$$|x|^3 < \epsilon.$$

We can take the cube root of both sides because the cube root function $g(x) = \sqrt[3]{x}$ is an increasing function, so if $|x|^3 < \epsilon$ then

$$\begin{aligned} g\left(|x|^3\right) &< g(\epsilon) \\ \sqrt[3]{|x|^3} &< \sqrt[3]{\epsilon} \\ |x| &< \sqrt[3]{\epsilon} \end{aligned}$$

Since $|x - 0| = |x|$, this suggests that setting $\delta = \sqrt[3]{\epsilon}$ is a good choice for δ. Note that since $\epsilon > 0$, $\delta = \sqrt[3]{\epsilon} > 0$. In fact, suppose

$$0 < |x - 0| < \delta = \sqrt[3]{\epsilon}.$$

Then

$$|x| < \sqrt[3]{\epsilon}.$$

We can cube both sides because the cube function $h(x) = x^3$ is an increasing function, so if $|x| < \sqrt[3]{\epsilon}$ then

$$\begin{aligned} h(|x|) &< h((\sqrt[3]{\epsilon}) \\ |x|^3 &< \sqrt[3]{\epsilon}^3 \\ |x^3| &< \epsilon & \text{because } |x|^3 = |x^3| \\ |x^3 - 0| &< \epsilon \end{aligned}$$

We have shown that for every $\epsilon > 0$, $\delta = \sqrt[3]{\epsilon} > 0$ is such that if $0 < |x - 0| < \delta$ then $|x^3 - 0| < \epsilon$.

3. (10 pts) Use the Limit Laws to find the exact value of

$$\lim_{x \to 0} \frac{\sqrt{3+x} - \sqrt{3}}{x}.$$

Do not forget to justify your work by referring to the Limit Laws and any other result you use.

We cannot start by using Limit Law #5 because we will end up with a limit in the denominator that is 0. So we will multiply the numerator and the denominator by $\sqrt{3+x} + \sqrt{3}$, which we can do because $\sqrt{3+x} + \sqrt{3} \geq \sqrt{3}$, so it is never 0.

$$\begin{aligned} \frac{\sqrt{3+x} - \sqrt{3}}{x} &= \frac{\sqrt{3+x} - \sqrt{3}}{x} \frac{\sqrt{3+x} + \sqrt{3}}{\sqrt{3+x} + \sqrt{3}} \\ &= \frac{\sqrt{3+x}^2 - \sqrt{3}^2}{x(\sqrt{3+x} + \sqrt{3})} \\ &= \frac{3 + x - 3}{x(\sqrt{3+x} + \sqrt{3})} \\ &= \frac{x}{x(\sqrt{3+x} + \sqrt{3})} \end{aligned}$$

We can now cancel x because $x \to 0$ means x is a number close to 0 but not equal to 0. So

$$\begin{aligned} \lim_{x \to 0} \frac{\sqrt{3+x} - \sqrt{3}}{x} &= \lim_{x \to 0} \frac{1}{\sqrt{3+x} + \sqrt{3}} \\ &= \frac{\lim_{x \to 0} 1}{\lim_{x \to 0} (\sqrt{3+x} + \sqrt{3})} & \text{by Limit Law \#5} \\ &= \frac{1}{\lim_{x \to 0} \sqrt{3+x} + \lim_{x \to 0} \sqrt{3}} & \text{by Limit Laws \#7 and \#1} \\ &= \frac{1}{\sqrt{\lim_{x \to 0} (3+x)} + \sqrt{3}} & \text{by Limit Laws \#11 and \#7} \\ &= \frac{1}{\sqrt{3+0} + \sqrt{3}} & \text{by the Direct Substitution Property} \\ &= \frac{1}{2\sqrt{3}} \end{aligned}$$

4. (5 pts each)

(a) Let S and T be nonempty sets. State the definition of a function f from S to T. What are the domain and the codomain of f?

A function $f : S \to T$ is a rule that assigns to each element $x \in S$ one and only one element $f(x)$ in T.

(b) Let S be the set of all gifts in Santa's bag, including those lumps of coal for the naughty children. Let T be the set of all elves that are employed at Santa's North Pole factory. Let $e : S \to T$ be the rule

$$e(x) = \text{the elf that wrapped } x.$$

Is e a function?

Whether e is a function is up to interpretation. Assuming that only elves work wrapping gifts in Santa's factory and each gift is wrapped by exactly one elf, e is a function because it assigns to each gift in Santa's bag one and only one elf that wrapped it.
On the other hand, if the elves collaborate wrapping gifts, then a gift may be wrapped by several elves and then e would not assign to that gift only one elf, so e would not be a function. Also, if there are any gifts in Santa's bag that are unwrapped–perhaps the lumps of coal–, or if anyone other than the elves is wrapping gifts in Santa's factory, e.g. Santa himself, or Mrs. Claus, or the reindeer–although wrapping gifts with hooves may be quite challenging–then e may not assign an elf to a particular gift x at all, so e is not a function.

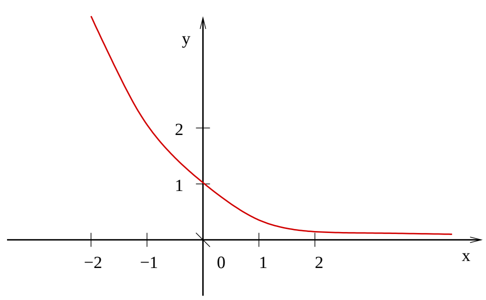
5. (5 pts each)

(a) Give an example of a function that is neither even nor odd. Don't forget to justify your example, that is explain why it is neither even nor odd.

Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x + 1$. Then $f(2) = 3$ and $f(-2) = -1$. So f cannot be an even function because $f(-2) \neq f(2)$, and f cannot be an odd function because $f(-2) \neq -f(2)$.

(b) Give an example of a nonlinear function that is decreasing throughout its domain. Again, remember to justify your example.

Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = (1/2)^x$. The graph of f below shows that if $x_1 < x_2$ then $f(x_1) > f(x_2)$, so f is a decreasing function. Since f is not of the form $mx + b$, it is not a linear function.

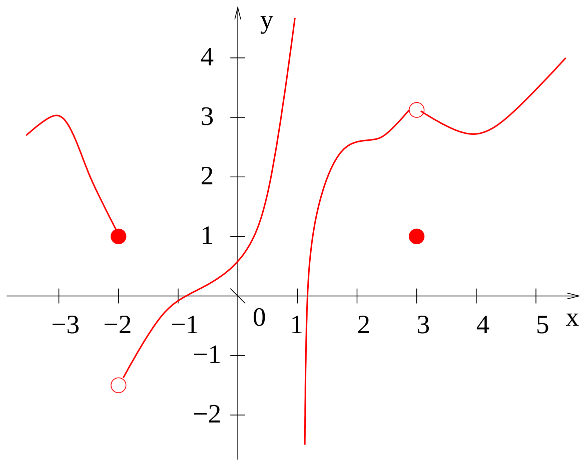


6. (a) (4 pts) Define what a continuous function is.

A function f is continuous at a number $x = a$ if

$$\lim_{x \to a} f(x) = f(a).$$

(b) (6 pts) At what numbers is the function f whose graph is given below discontinuous? Identify what type of discontinuity f has at each such number.



The function f has a

- jump discontinuity at $x = -2$ because the one-sided limits $\lim_{x\to -2^-} f(x)$ and $\lim_{x\to -2^-} f(x)$ both exist but are not equal,
- an infinite discontinuity at $x = 1$ because the values of f get arbitrarily large as $x \to 1^-$ or arbitrarily negative as $x \to 1^+$,
- a removable discontinuity at $x = 3$ because the limit $\lim_{x\to 3} f(x) = 3$ exists but is not equal to $f(3) = 1$.

7. **Extra credit problem.**

(a) (10 pts) We noted in class that

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$$

does not exist. Use the $\delta - \epsilon$ definition of a limit to prove this. Hint: suppose the limit exists and is L, and show that for $\epsilon = 1/2$ there no $\delta > 0$ that would satisfy the $\delta - \epsilon$ definition of a limit.

Suppose

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) = L.$$

This is true if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $0 < |x - 0| < \delta$ then

$$\left|\sin\left(\frac{1}{x}\right) - L\right| < \epsilon.$$

In particular, there should be such a δ for $\epsilon = 1/2$. When $\epsilon = 1/2$,

$$\left|\sin\left(\frac{1}{x}\right) - L\right| < \epsilon.$$

is the same thing as

$$L - \frac{1}{2} < \sin\left(\frac{1}{x}\right) < L + \frac{1}{2}.$$

But no matter how small δ is, it is always possible to find $-\delta < x < \delta$ such that $\sin(1/x) = 1$. For this, all we need is that $1/x = \pi/2 + 2k\pi$. That is

$$x = \frac{1}{\pi/2 + 2k\pi}.$$

If we make k large, then $\frac{1}{\pi/2+2k\pi}$ will be small. By making k sufficiently large, we can make $\frac{1}{\pi/2+2k\pi}$ as small as we want. In fact, we can solve the inequality

$$\frac{1}{\pi/2 + 2k\pi} < \delta$$

by multiplying both sides by $\pi/2+2k\pi$. If k is positive, then $\pi/2+2k\pi$ is also positive, so the direction of the inequality stays the same when we multiply both sides by $\pi/2+2k\pi$:

$$\begin{aligned}
\frac{1}{\pi/2 + 2k\pi} &< \delta \\
1 &< \left(\frac{\pi}{2} + 2k\pi\right)\delta \\
\frac{1}{\delta} &< \frac{\pi}{2} + 2k\pi && \text{note } \delta > 0 \\
\frac{1}{\delta} - \frac{\pi}{2} &< 2k\pi \\
\frac{\frac{1}{\delta} - \frac{\pi}{2}}{2\pi} &< k \\
\frac{1}{2\pi\delta} - \frac{1}{4\pi} &< k
\end{aligned}$$

Whatever number is on the left-hand side of the inequality, it is always possible to choose a positive integer k larger than that number. For such a k, the number $x_1 = \frac{1}{\pi/2+2k\pi}$ satisfies

$$0 < x_1 < \delta \implies |x_1 - 0| < \delta$$

while

$$\sin\left(\frac{1}{x_1}\right) = \sin\left(\frac{\pi}{2} + 2k\pi\right) = 1.$$

Now, let $x_2 = \frac{1}{3\pi/2+2k\pi}$. Then $x_2 < x_1$ since $3\pi/2 + 2k\pi > \pi/2 + 2k\pi$. Hence it is also true that

$$0 < x_2 < \delta \implies |x_2 - 0| < \delta$$

while

$$\sin\left(\frac{1}{x_2}\right) = \sin\left(\frac{3\pi}{2} + 2k\pi\right) = -1.$$

Now, it cannot be that

$$\begin{aligned}
L - \frac{1}{2} < \sin\left(\frac{1}{x_1}\right) < L + \frac{1}{2} \\
L - \frac{1}{2} < \sin\left(\frac{1}{x_2}\right) < L + \frac{1}{2}
\end{aligned}$$

are both true because there is no way that both -1 and 1 are between $L - 1/2$ and $L + 1/2$ on the number line. So there is no value of $\delta > 0$, however small that it would be true for every x that if $0 < |x - 0| < \delta$ then $|\sin(1/x) - L| < \epsilon$. Therefore L cannot be the limit of $\sin(1/x)$ as $x \to 0$. This argument works for any value of L. Therefore

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

does not exist.

(b) (5 pts) Would a similar argument show that

$$\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$$

does not exist? Explain your reasoning.

Yes, the same kind of argument would work, except $\cos(1/x)$ would be 1 when $x_1 = \frac{1}{2k\pi}$ and -1 when $x_2 = \frac{1}{\pi + 2k\pi}$. By choosing k large enough, we can find such and x_1 and x_2 as close as we want to 0.