MCS 118 Exam 1 Solutions

1. (10 pts) Determine if the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x|x|$$

is even, odd, or neither and justify your answer. If you have a graphing calculator, you can check your answer visually, or even get an idea of what the answer should be, but the apparent symmetry of a graph on the screen of a graphing calculator is not good evidence of evenness or oddness.

First, notice that $|-x| = |x|$ for any $x \in \mathbb{R}$ since $-x$ and x are the same distance from 0 on the number line. Now, for any $x \in \mathbb{R}$,

$$f(-x) = -x|-x| = -x|x| = -f(x).$$

Hence f is an odd function.

2. The monthly cost of driving a car depends on the number of miles driven. Lynn found that in May, it cost her \$380 to drive 480 miles and in June it cost her \$460 to drive 800 miles.

(a) (4 pts) Express the monthly cost C as a function of the distance driven d assuming that a linear relationship gives a suitable model.

So $C(d) = md + b$, where m is the slope and b is the vertical intercept. As usual,

$$m = \frac{\Delta C}{\Delta d} = \frac{460 - 380}{800 - 480} = \frac{80}{320} = \frac{1}{4} \frac{\$}{\text{mile}}.$$

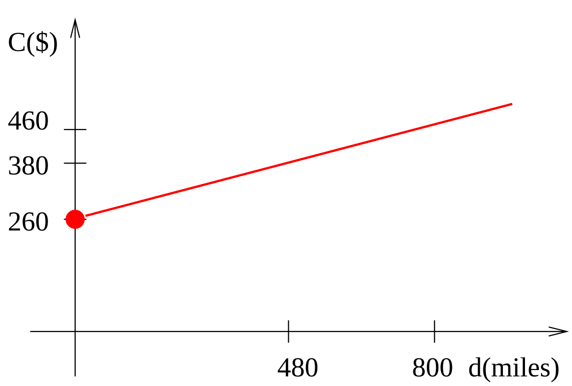
Now that we know $C = d/4 + b$, we can find the value of b by substituting either of the given points:

$$460 = \frac{800}{4} + b \implies b = 460 - 200 = \$260.$$

So

$$C(d) = \frac{d}{4} + 260.$$

(b) (4 pts) Draw the graph of the linear function. What does the slope represent? (Hint: what are the units of the slope?)



The slope, which is 1/4 \$/mile, is the cost of each additional mile driven.

(c) (2 pts) What does the vertical intercept represent?

It is the cost of having the car per month if it is not driven at all. That is the fixed monthly cost. In practice, this may consist of car insurance, a loan payment, property tax or registration fees, etc.

3. (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = |3-x||x+1|$. Express f as a piecewise defined function. (Hint: you do not necessarily need to consider four cases here; you can get away with only two.)

We solved such a problem in class by splitting it into four cases depending on the signs of the quantities inside the absolute values. Here is how that would go.

(a) Case $3 - x \geq 0$ and $x + 1 \geq 0$. Then

$$3 - x \geq 0 \implies 3 \geq x$$
$$x + 1 \geq 0 \implies x \geq -1$$

So $-1 \leq x \leq 3$. In this case,

$$f(x) = |3-x||x+1| = (3-x)(x+1) = -x^2 + 2x + 3.$$

(b) Case $3 - x \geq 0$ and $x + 1 < 0$. Then

$$3 - x \geq 0 \implies 3 \geq x$$
$$x + 1 < 0 \implies x < -1$$

Since $x \leq 3$ and $x < -1$ must both hold, this is the case if $x < -1$. If so,

$$f(x) = |3-x||x+1| = (3-x)(-x-1) = x^2 - 2x - 3.$$

(c) Case $3 - x < 0$ and $x + 1 \geq 0$. Then

$$3 - x < 0 \implies 3 < x$$
$$x + 1 \geq 0 \implies x \geq -1$$

Since $x > 3$ and $x \geq -1$ must both hold, this is the case if $x > 3$. If so,

$$f(x) = |3-x||x+1| = (-3+x)(x+1) = x^2 - 2x - 3.$$

(d) Case $3 - x < 0$ and $x + 1 < 0$. Then

$$3 - x < 0 \implies 3 < x$$
$$x + 1 < 0 \implies x < -1$$

But $x > 3$ and $x < -1$ must both hold, which is impossible. So there is no real number x that fits this case.

We can now conclude that

$$f(x) = \begin{cases} -x^2 + 2x + 3 & \text{if } -1 \leq x \leq 3 \\ x^2 - 2x - 3 & \text{if } x < -1 \text{ or } x > 3 \end{cases}.$$

Alternately, you can do this by looking at two cases only. First, notice that

$$f(x) = |3-x||x+1| = |(3-x)(x+1)|.$$

What the absolute value does depend on the sign of $(3-x)(x+1) = -x^2 + 2x + 3$. Notice that the graph of this quadratic funtion is an upside-down parabola since the coefficient of x^2 is negative. Hence $(3-x)(x+1)$ is positive between the two places where it crosses the x-axis and negative outside those two places. Since it is already factored, it is easy to see that its roots are $x = -1$ and $x = 3$. These are the two places where it crosses the x-axis. Hence

(a) Case $(3-x)(x+1) \geq 0$. As we noted, this is the case when $-1 \leq x \leq 3$. In this case,

$$f(x) = |(3-x)(x+1)| = (3-x)(1+x) = -x^2 + 2x + 3.$$

(b) Case $(3-x)(x+1) < 0$. As we noted, this is the case when $x < -1$ or $3 < x$. In this case,

$$f(x) = |(3-x)(x+1)| = -(3-x)(1+x) = x^2 - 2x - 3.$$

So we have once again obtained

$$f(x) = \begin{cases} -x^2 + 2x + 3 & \text{if } -1 \leq x < -1 \leq 3 \\ x^2 - 2x - 3 & \text{if } x < -1 \text{ or } x > 3 \end{cases}.$$

4. (5 pts each)

(a) Give an example of a polynomial $p(x)$ of degree 3 whose value is 0 at $x = -3$, $x = 2$, and $x = 5$ and whose value is 36 at $x = 3$.

Since $p(-3) = 0$, $p(2) = 0$, and $p(5) = 0$ is 0, we know that $x+3$, $x-2$, and $x-5$ must be factors of $p(x)$. Multiplying these three factors already gives x^3, so $p(x)$ cannot have any additional factors that include x. So the only remaining factor must be a constant number:

$$p(x) = a(x+3)(x-2)(x-5).$$

We can now find a from $p(3) = 36$:

$$36 = a(3+3)(3-2)(3-5) = -12a \implies a = -3.$$

So

$$p(x) = -3(x+3)(x-2)(x-5).$$

Actually, this is the only example of such a polynomial.

(b) Define what a rational function is and give an example of a rational function whose domain is all real numbers except $x = 4$. Do not forget to justify your example.

A rational function is a polynomial divided by a nonzero polynomial. That is $p(x)/q(x)$ where p and q are polynomials and q is not the zero polynomial.
The function $f(x) = \frac{x^2+3}{x-4}$ is an example of a rational function whose domain is $\mathbb{R} \setminus \{4\}$. First, notice that the numerator and the denominator are both polynomials. Second, the denominator is 0 at exactly $x = 4$ and nonzero elsewhere. So $x = 4$ cannot be in the domain, but any other real number can be.

5. (10 pts) **Extra credit problem.** Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function such that $f(x) \neq 0$ for any real number x. Is the function $g(x) = f(-x)$ always decreasing? If you think it is, give a proof; if you do not think it is, find a counterexample.

Yes, g must be decreasing. Let $x_1 < x_2$. We can multiply both sides of this inequality by -1 to get $-x_1 > -x_2$. We know $f(-x_1) > f(-x_2)$ because f is an increasing function. Hence

$$g(x_1) = f(-x_1) > f(-x_2) = g(x_2).$$

This must always be true if $x_1 < x_2$, which shows g is a decreasing function.