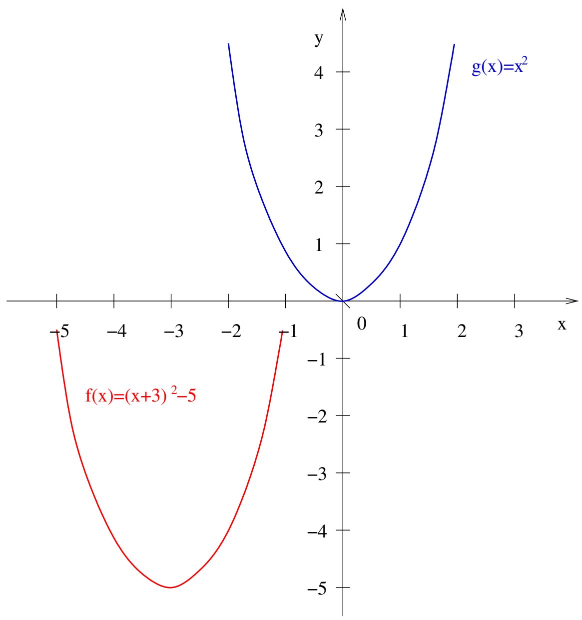
MCS 118 Exam 2 Solutions

1. (10 pts) Graph the function $f(x) = x^2+6x+4$ by hand, not by plotting points, but by starting with the graph of a standard power function and then applying appropriate transformations.

   First, let us complete the square:
   $$f(x) = x^2 + 6x + 4 = x^2 + 6x + 9 - 9 + 4 = (x+3)^2 - 5.$$
   Notice that if $g(x) = x^2$ then $f(x) = g(x+3) - 5$. So we can start with the parabola that is the graph of a g and then we shift it left by 3 units and down by 5 units to get the graph of f.



2. (5 pts each) Let $f(x) = \sqrt{x}$ and $g(x) = \sqrt[3]{1-x}$.
   (a) Find $g \circ f(x)$ and the largest possible subset of the real numbers that could be the domain of $g \circ f$.
   $$g \circ f(x) = g(f(x)) = g(\sqrt{x}) = \sqrt[3]{1-\sqrt{x}}$$
   For $x \in \mathbb{R}$ to be in the domain, $g \circ f(x)$ must have a value that is a real number. For $\sqrt{x}$ to have a real value x must be nonnegative. Since any real number has a cube root, this is the only restriction on x. So the domain of $g \circ f$ is $\mathbb{R}^{\geq 0}$.

   (b) Find $f \circ g(x)$ and the largest possible subset of the real numbers that could be the domain of $f \circ g$.
   $$f \circ g(x) = f(g(x)) = f(\sqrt[3]{1-x}) = \sqrt{\sqrt[3]{1-x}}$$
   Any real number has a cube root. But the number under the square root must be nonnegative for $f \circ g(x)$ to have a real value. So we want $\sqrt[3]{1-x} \geq 0$. The cube root of a number is nonnegative if the number itself is nonnegative. So we want
   $$1 - x \geq 0 \implies 1 \geq x$$

Hence the domain of $f \circ g$ is $(-\infty, 1]$.

3. (a) (4 pts) Let f be a function of real numbers and a and L real numbers. State the definition (informal or formal, your choice) of

$$\lim_{x \to a} f(x) = L.$$

We say the limit of $f(x)$ as x approaches a is L if the value of $f(x)$ can be made to get arbitrarily close to L for values of x that are sufficiently close to a.

A note on definitions: definitions are fundamental organizing blocks of mathematics. They ensure that everybody studying some mathematical object has the same understanding of what such an object is. You can state a definition using your own words. For example, you can say "as close as you want" instead of "arbitrarily close", but you must take care that the meaning remains the same as the accepted definition. E.g. if you switch arbitraty and sufficient in the above definition, the meaning changes and it is no longer the correct definition of limit. You can rephrase but not reinvent a definition.

(b) (6 pts) Give an example of a function f and a number a such that $\lim_{x \to a} f(x)$ does not exist. Be sure to justify your example.

One example we gave in class was $f(x) = \sin(\pi/x)$ as $x \to 0$. Notice that f has no value if $x = 0$ but does have a value for every x near 0. The reason

$$\lim_{x \to 0} \sin(\pi/x)$$

does not exist is that as x gets close to 0, π/x becomes a really large or really negative number. As this happens, the angle π/x keeps running around the unit circle, with the corresponding value of $\sin(\pi/x)$ oscillating between -1 and 1. In fact, the closer x gets to 0, the faster the value of π/x changes and hence the faster $\sin(\pi/x)$ oscillates between -1 and 1. In any case, there is no particular number the values of $\sin(\pi/x)$ get close to.

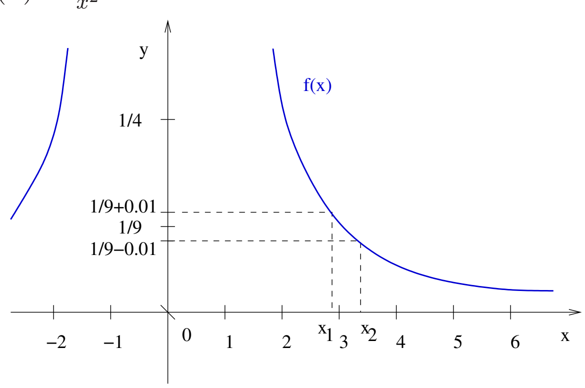
4. (10 pts) Let $f(x) = \frac{1}{x^2}$. Show that the value of $f(x)$ can be made to be within 0.01 of 1/9 if x is sufficiently close to 3 by finding two numbers $x_1 < 3 < x_2$ such that if $x_1 < x < x_2$ then

$$\frac{1}{9} - 0.01 < f(x) < \frac{1}{9} + 0.01.$$

So we want $f(x) = \frac{1}{x^2}$ to be between $\frac{1}{9} - 0.01$ and $\frac{1}{9} + 0.01$:

$$\frac{1}{9} - 0.01 < \frac{1}{x^2} < \frac{1}{9} + 0.01.$$

Here is the graph of $f(x) = \frac{1}{x^2}$:



This shows that the two numbers x needs to be between are x_1 and x_2 such that

$$f(x_1) = \frac{1}{{x_1}^2} = \frac{1}{9} + 0.01 = \frac{1}{9} + \frac{1}{100} = \frac{100}{900} + \frac{9}{900} = \frac{109}{900}$$
$$f(x_2) = \frac{1}{{x_2}^2} = \frac{1}{9} - 0.01 = \frac{1}{9} - \frac{1}{100} = \frac{100}{900} - \frac{9}{900} = \frac{91}{900}$$

So

$$\frac{1}{{x_1}^2} = \frac{109}{900} \implies {x_1}^2 = \frac{1}{\frac{109}{900}} = \frac{900}{109} \implies x_1 = \sqrt{\frac{900}{109}}$$

and

$$\frac{1}{{x_2}^2} = \frac{91}{900} \implies {x_2}^2 = \frac{1}{\frac{91}{900}} = \frac{900}{91} \implies x_2 = \sqrt{\frac{900}{91}}$$

So if x is close enough to 3 that

$$\sqrt{\frac{900}{109}} < x < \sqrt{\frac{900}{91}}$$

then

$$\frac{1}{9} - 0.01 < \frac{1}{x^2} < \frac{1}{9} + 0.01.$$

We can also use decimal approximations to these numbers, but be careful to round the lower bound up and the upper bound down to make sure that any x that is between the decimal approximations is still between $\sqrt{\frac{900}{109}}$ and $\sqrt{\frac{900}{91}}$. E.g.

$$2.874 < x < 3.144$$

would work.

5. (5 pts each) **Extra credit problem.** Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions.
   (a) Show that if f is increasing and $g : \mathbb{R} \to \mathbb{R}$ is decreasing then the composition $f \circ g$ is always a decreasing function. (Remember that an example does not prove that $f \circ g$ is always a decreasing function no matter what increasing function f is and what decreasing function g is.)

   We need to show that for any $x_1 < x_2$, $g \circ f(x_1) > g \circ f(x_2)$. So let $x_1 < x_2$. We know $g(x_1) > g(x_2)$ because g is decreasing. Now, f is an increasing function, so the larger input to f will result in the larger output. So $f(g(x_1)) > f(g(x_2))$. This is what we wanted to show.

   (b) What if f and g are both decreasing? Does $f \circ g$ have to be increasing or decreasing or can it be neither in this case?

   In this case, $f \circ g$ will be increasing. Let $x_1 < x_2$. Then $g(x_1) > g(x_2)$ as before. Now, f is a decreasing function, so the larger input to f will result in the smaller output. So $f(g(x_1)) < f(g(x_2))$. Hence $f \circ g$ is an increasing function.