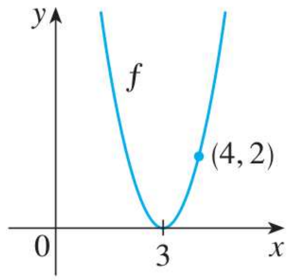
MCS 118 Final Exam Solutions

Dec 18, 2019

1. (10 pts) Find the expression for the quadratic function whose graph is shown. Be sure to carefully explain (in words) how you got your answer.



First, notice that f is a polynomial which has a zero at $x = 3$. So $x - 3$ is a factor of f. In fact, $x = 3$ is the only zero of f, so $x - 3$ is the only factor of f. Since f is of degree 2, we need two factors of $x - 3$. That is $f(x) = a(x-3)^2$ for some $a \in \mathbb{R}$. Now

$$2 = f(4) = a(4-3)^2 = a,$$

so $f(x) = 2(x-3)^2$.

2. (10 pts) Prove the statement with the $\delta - \epsilon$ definition of a limit:

$$\lim_{x \to -2} (3x + 5) = -1.$$

We need to show that for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - (-1)| < \epsilon$$

whenever

$$0 < |x - (-2)| < \delta.$$

So we want

$$|(3x + 5) - (-1)| < \epsilon.$$

Notice that

$$|(3x + 5) - (-1)| = |3x + 6| = |3(x + 2)| = |3||x + 2| = 3|x + 2|.$$

This suggests that we should make $|x + 2| < \epsilon/3$.

So let $\delta = \epsilon/3$ and $0 < |x + 2| < \delta = \epsilon/3$. Then

$$|(3x + 5) - (-1)| = |3x + 6| = |3(x + 2)| = |3||x + 2| = 3|x + 2| < 3\frac{\epsilon}{3} = \epsilon,$$

which is what we wanted to show.

3. (10 pts) Use the definition of continuity and the properties of limits to show that the function

$$f(x) = 3x^4 - 5x + \sqrt[3]{x^2 + 4}$$

is continuous at 2.

We need to show

$$\lim_{x \to 2} f(x) = f(2).$$

Note that

$$\begin{aligned}
\lim_{x\to 2} f(x) &= \lim_{x\to 2}(3x^4 - 5x + \sqrt[3]{x^2+4}) \\
&= \lim_{x\to 2}(3x^4 - 5x) + \lim_{x\to 2}\sqrt[3]{x^2+4} && \text{by LL1} \\
&= 3(2^4) - 5(2) + \lim_{x\to 2}\sqrt[3]{x^2+4} && \text{since } 3x^4 - 5x \text{ is a polynomial} \\
&= 3(2^4) - 5(2) + \sqrt[3]{\lim_{x\to 2}(x^2+4)} && \text{by LL11} \\
&= 3(2^4) - 5(2) + \sqrt[3]{2^2+4} && \text{since } x^2+4 \text{ is a polynomial} \\
&= f(2),
\end{aligned}$$

which is what we wanted to show.

4. (5 pts each)
   (a) Define what even and odd functions are.

   The function f is even if $f(-x) = f(x)$ for every x in its domain, and odd if $f(-x) = -f(x)$ for every x in its domain.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be an even function and $g : \mathbb{R} \to \mathbb{R}$ be an odd function such that $g(x) \neq 0$. What kind of symmetry (even, odd, neither?) does f/g have in general? (Remember that one specific example is not sufficient evidence to prove a general statement.)

   We will show that f/g is odd in this case. Since $f(-x) = f(x)$ and $g(-x) = -g(x)$ for every x,

$$(f/g)(-x) = \frac{f(-x)}{g(-x)} = \frac{f(x)}{-g(x)} = -\frac{f(x)}{g(x)} = -(f/g)(x)$$

   for every x. Hence f/g is odd.

5. (10 pts) Find

$$\lim_{x\to 0^+}\left[\cos\left(\frac{1-x}{x}\right)\sin(x)\right].$$

(Hint: Refer to the illustration on the left. Use the Sandwich (Squeeze) Theorem.)

First, note that for any $x \neq 0$,

$$-1 \leq \cos\left(\frac{1-x}{x}\right) \leq 1.$$

Also, since $x \to 0^+$, we may assume that x is a positive number close to 0, so $\sin(x) > 0$. Therefore we can multiply the inequality above by $\sin(x)$, and its direction does not change:

$$-\sin(x) \leq \cos\left(\frac{1-x}{x}\right)\sin(x) \leq \sin(x).$$

for every $x > 0$ near 0. We saw in class that

$$\lim_{x\to 0}\sin(x) = 0.$$

It follows by LL3 that

$$\lim_{x\to 0} -\sin(x) = -\lim_{x\to 0}\sin(x) = -0 = 0.$$

In particular,

$$\lim_{x\to 0^+}\sin(x) = 0 \qquad \text{and} \qquad \lim_{x\to 0^+} -\sin(x) = 0.$$

So by the Squeeze Theorem,

$$\lim_{x\to 0^+}\left[\cos\left(\frac{1-x}{x}\right)\sin(x)\right] = 0.$$

6. (5 pts each) Let

$$f(x) = \frac{x^2+2x-8}{|x-2|}.$$

   (a) Find

$$\lim_{x\to 2} f(x)$$

   if it exists. If the limit does not exist, explain why it does not exist.

   Note that $x^2+2x-8 = (x+4)(x-2)$. So

$$f(x) = \frac{(x+4)(x-2)}{|x-2|}.$$

   Now, if $x > 2$, then $x-2 > 0$, so $|x-2| = x-2$ and

$$f(x) = \frac{(x+4)(x-2)}{|x-2|} = \frac{(x+4)(x-2)}{x-2} = x+4,$$

   where we could cancel $x-2$ because it is not 0. Hence

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(x+4) = 2+4 = 6$$

   by direct substitution since $x+4$ is a polynomial.
   On the other hand, if $x < 2$, then $x-2 < 0$, so $|x-2| = -(x-2)$ and

$$f(x) = \frac{(x+4)(x-2)}{|x-2|} = \frac{(x+4)(x-2)}{-(x-2)} = -(x+4) = -x-4,$$

   where we could cancel $x-2$ because it is not 0. Hence

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(-x-4) = -2-4 = -6$$

   by direct substitution since $-x-4$ is a polynomial.
   Since the one-sided limits both exist but are not equal, the regular limit does not exist.

   (b) Find

$$\lim_{x\to -4} f(x)$$

   if it exists. If the limit does not exist, explain why it does not exist.

   If x is a number close to -4, then $x-2$ is close to -6. In particular, $x-2$ is negative, so $|x-2| = -(x-2)$. Therefore

$$f(x) = \frac{(x+4)(x-2)}{|x-2|} = \frac{(x+4)(x-2)}{-(x-2)} = -(x+4) = -x-4,$$

   for values of x close to -4. Hence

$$\lim_{x\to -4} f(x) = \lim_{x\to -4}(-x-4) = 4-4 = 0$$

   by direct substitution since $-x-4$ is a polynomial.

7. **Extra credit problem.** Let f and g be functions and a a real number such that

$$\lim_{x\to a} f(x) = L_1 \qquad \text{and} \qquad \lim_{x\to a} g(x) = L_2$$

both exist.

(a) (10 pts) Prove that if $f(x) \leq g(x)$ for every x is some neighborhood of a except possibly at $x = a$, then $L_1 \leq L_2$. (Hint: Suppose that $L_1 \leq L_2$ is not true, that is $L_2 < L_1$. Now use the definition of the limit with $\epsilon = (L_1 - L_2)/2$ to show that it cannot be true that $f(x) \leq g(x)$ for every x near a.)

Suppose that $L_1 \leq L_2$ is not true, that is $L_2 < L_1$ is true. We will show that this leads to a contradiction. Since $L_1 > L_2$, we know $L_1 - L_2 > 0$. Let $\epsilon = (L_1 - L_2)/2$, which is a positive number. Since

$$\lim_{x \to a} f(x) = L_1$$

there is a $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$ then $|f(x) - L_1| < \epsilon$. That is

$$L_1 - \epsilon < f(x) < L_1 + \epsilon.$$

Hence

$$f(x) > L_1 - \epsilon > L_1 - \frac{L_1 - L_2}{2} = \frac{2L_1 - L_1 + L_2}{2} = \frac{L_1 + L_2}{2}.$$

Similarly, there is a $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$ then $|g(x) - L_2| < \epsilon$. That is

$$L_2 - \epsilon < g(x) < L_2 + \epsilon.$$

Hence

$$g(x) < L_2 + \epsilon < L_2 + \frac{L_1 - L_2}{2} = \frac{2L_2 + L_1 - L_2}{2} = \frac{L_1 + L_2}{2}.$$

Finally, the problem states that there is some $\delta_3 > 0$ such that if $0 < |x - a| < \delta_3$ then $f(x) \leq g(x)$. Now, let $\delta = \min(\delta_1, \delta_2, \delta_3)$ and suppose $0 < |x - a| < \delta$. Then

$$\begin{aligned} 0 &< |x - x_0| < \delta_1 \\ 0 &< |x - x_0| < \delta_2 \\ 0 &< |x - x_0| < \delta_3 \end{aligned}$$

are all true. Hence

$$g(x) < \frac{L_1 + L_2}{2} < f(x)$$

and

$$f(x) \leq g(x)$$

must both be true. But $g(x) < f(x)$ and $f(x) \leq g(x)$ contradict each other. Therefore the assumption that $L_1 > L_2$ cannot be correct. That is, $L_1 \leq L_2$ must always be true.

(b) (5 pts) Now suppose that $f(x) < g(x)$ for every x is some neighborhood of a except possibly at $x = a$. Does it have to be true that $L_1 < L_2$? If you think it is true, give a proof; if you do not think it is true, find a counterexample.

This is not true. For example, if $f(x) = x^2$ and $g(x) = 2x^2$, then it is true for any $x \neq 0$ that

$$0 < x^2 \implies x^2 < 2x^2 \implies f(x) < g(x).$$

But

$$\begin{aligned} L_1 &= \lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 = 0^2 = 0 \\ L_2 &= \lim_{x \to 0} g(x) = \lim_{x \to 0} (2x^2) = 2(0^2) = 0 \end{aligned}$$

by the direct substitution property of polynomials. Thus, it need not be true that $L_1 < L_2$.