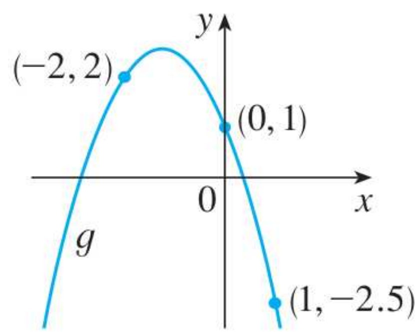
MCS 118 Exam 1 Solutions

1. (10 pts) Express the surface area of a cube as a function of its volume.

   Let x be the side of the cube. Since the surface of the cube consists of six squares, its surface ares is $SA = 6x^2$. The volume is $V = x^3$. Taking cube roots of both sides in the latter allows us to express the side in terms of the volume: $x = \sqrt[3]{V}$. We can now substitute this into SA to get

   $$SA(V) = 6\sqrt[3]{V}^2.$$

2. (10 pts) Find an expression for the quadratic polynomial function whose graph is shown.



   We know g is a quadratic polynomial, so it must be of the form $g(x) = ax^2 + bx + c$. Since the graph passes through $(-2, 2)$, $(0, 1)$, and $(1, -2.5)$,

   $$\begin{aligned} 2 &= g(-2) = a(-2)^2 + b(-2) + c = 4a - 2b + c \\ 1 &= g(0) = a0^2 + b0 + c = c \\ -2.5 &= g(1) = a1^2 + b1 + c = a + b + c \end{aligned}$$

   The second equation immediately gives $c = 1$. Substituting this into the other equations yields

   $$\begin{aligned} 4a - 2b + 1 = 2 &\implies 4a - 2b = 1 \\ a + b + 1 = -2.5 &\implies a + b = -3.5. \end{aligned}$$

   Multiply the second equation by 2 to get $2a + 2b = -7$, then add the two equations to get $6a = -6$, and hence $a = -1$. Now $b = -3.5 - a = -2.5$. So

   $$g(x) = -x^2 - 2.5x + 1.$$

   We can that this is correct by substituting the three points:

   $$\begin{aligned} g(-2) &= -(-2)^2 - 2.5(-2) + 1 = 2 && \checkmark \\ g(0) &= -0^2 - 2.5(0) + 1 = 1 && \checkmark \\ g(1) &= -1^2 - 2.5(1) + 1 = -2.5 && \checkmark \end{aligned}$$

3. (a) (4 pts) Let S and T be nonempty sets. State the definition of a function $f : S \to T$.

   Here is what the book says: a *function* is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

Here is how I wrote it in the notes posted on the class website: a *function* f from a set S to a set T is a rule which assigns to every element x in S one and only one element $f(x)$ in T.
Here is how we stated this in class: a *function* from a set S to a set T is a rule which assigns to every element x in S one and only one element $f(x)$ in T.

(b) (3 pts)

The crowd gathers at Moe's Tavern on a Friday night. Let S be the set of all people in the bar and let T be the set of all bottles in the bar containing beverages. Construct a rule $f : S \to T$ that is a function and carefully explain why f is a function.

To make the setup more mathematically precise, we will need to make some assumptions about the people in the bar. We will make the (quite reasonable) assumptions that all the people in the tavern (including Moe, the bartender) have at some point had an adult beverage at Moe's other than draft and canned beer, and that Moe regularly stocks the same kinds of beverages in his bar.
Let

$$f(x) = \text{the bottle whose content } x \text{ most recently drank from.}$$

Now, every person x in the tavern has had some beverage other than draft and canned beer at some point at Moe's. Therefore x has also had a most recent drink. The contents of that beverage would have come from some bottle or bottles in the bar. The value of $f(x)$ is the last bottled ingredient that Moe would have added to that most recent drink. Actually, $f(x)$ is the bottle that contains that last bottled ingredient. By our assumptions, $f(x)$ must have a value for every person x in the bar, and by specifying that $f(x)$ is the last bottled ingredient in the most recent such drink, we have made sure that there is exactly one value of $f(x)$. Therefore f is a function.

(c) (3 pts) Using the same sets S and T, construct a rule $g : S \to T$ that is not a function and carefully explain why g is not a function.

Let us make the same assumptions as in part (b). Let

$$f(x) = \text{the bottle whose content } x \text{ has drunk from.}$$

This rule is not a function because it is quite certain that there is at least one person x in the tavern (if no one else, at the very least Moe himself) that has tried the contents of more than one bottle. Then $f(x)$ would have multiple values for such an x.

4. (10 pts) Find the largest possible subset of the real numbers that could be the domain of the function

$$f(x) = \sqrt[4]{5 - \frac{2}{x}}.$$

There are two reasons a real number x could not be in the domain of f: we cannot have 0 in the denominator of the fraction and the 4th root of a negative number is not a real number. So $x = 0$ is not in the domain because it would result in $2/0$. The number under the 4th root would be negative if

$$5 - \frac{2}{x} < 0$$

We can add $2/x$ to both side to get

$$5 < \frac{2}{x}$$

It is now tempting to multiply both sides by x. We know $x \neq 0$ already, but we need to be careful if we multiply the inequality by x because x could be positive or negative, and if x is negative, then multiplying by x would reverse the inequality. In fact, if x is negative, then $2/x$ is also negative, and a negative number cannot be greater than 5. Hence we can rule this possibility out. If x is positive, then

$$5x < 2 \implies x < \frac{2}{5}.$$

So the values of x that make $5 - 2/x$ negative are $0 < x < 2/5$. These cannot be in the domain and neither can $x = 0$. Therefore the largest possible domain consists of the numbers $x < 0$ and the numbers $2/5 \leq x$. So

$$D(f) = \left\{ x \in \mathbb{R} \mid x < 0 \text{ or } \frac{2}{5} \leq x \right\} = (\infty, 0) \cup \left[\frac{2}{5}, \infty\right).$$

5. (5 pts each) **Extra credit problem.**
   (a) Let f and g be functions $\mathbb{R} \to \mathbb{R}$. If f is an even function and fg is an odd function, is g an odd function?

   No, g does not need to be an odd function in this case. For example, let $f(x) = 0$ and $g(x) = x + 1$. Then f is even since $f(-x) = 0 = f(x)$ for every $x \in \mathbb{R}$, and $fg(x) = 0$ is odd since $fg(-x) = 0 = -fg(x)$ for every $x \in \mathbb{R}$. But g is not odd because $g(-x) = -x + 1 \neq -x - 1 = -g(x)$.

   (b) Let $\mathbb{R}^*$ denote the set of all nonzero real numbers and let f and g be functions $\mathbb{R} \to \mathbb{R}^*$. If f is an even function and fg is an odd function, is g an odd function?

   Yes, g must be odd in this case. This is because we can get g by dividing fg by f. Since the codomain of f is the set of nonzero real numbers, $f(x)$ is never 0, and so

   $$g(x) = \frac{fg(x)}{f(x)}$$

   for every $x \in \mathbb{R}$. As you saw on your Webwork homework (problem 6 in HW2), an odd function divided by an even function is an odd function in general. Here is why that is true in this case:

   $$g(-x) = \frac{fg(-x)}{f(-x)} = \frac{-fg(x)}{f(x)} = -\frac{fg(x)}{f(x)} = -g(x)$$

   for every $x \in \mathbb{R}$. Note that we used that fg is odd, so $fg(-x) = -fg(x)$ for every $x \in \mathbb{R}$, and f is even, so $f(-x) = 0 = f(x)$ for every $x \in \mathbb{R}$.