MCS 118 Exam 2 Solutions

1. (5 pts each) Let $f(x) = \sqrt{3-x}$ and $g(x) = \sqrt{x^2-1}$.

   (a) Find the function fg and the largest possible subset of the real numbers that could be the domain of fg.

   First,
   $$(fg)(x) = f(x)g(x) = \sqrt{3-x}\sqrt{x^2-1}.$$
   For a real number x to be in the domain, both $f(x) = \sqrt{3-x}$ and $g(x) = \sqrt{x^2-1}$ must be real numbers. The value of $\sqrt{3-x}$ is real whenever
   $$3 - x \geq 0 \iff 3 \geq x.$$
   The value of $\sqrt{x^2-1}$ is real whenever
   $$x^2 - 1 \geq 0.$$
   Note that $h(x) = x^2 - 1$ is a quadratic function in which the x^2 term has a positive coefficient, so its graph is a right-side up parabola. Therefore its value is negative between its two roots. Since $x^2 - 1 = (x+1)(x-1)$, its two roots are 1 and -1. So $x^2 - 1 < 0$ when $-1 < x < 1$, and hence $x^2 - 1 \geq 0$ when $x \leq -1$ or $1 \leq x$.
   Therefore $(fg)(x) = \sqrt{3-x}\sqrt{x^2-1}$ has a real value whenever $x \leq -1$ or $1 \leq x \leq 3$. So the largest possible domain of fg is
   $$D(fg) = (\infty, -1] \cup [1, 3].$$

   (b) Find the function f/g and the largest possible subset of the real numbers that could be the domain of f/g.

   Now,
   $$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{3-x}}{\sqrt{x^2-1}}.$$
   For a real number x to be in the domain, both $f(x) = \sqrt{3-x}$ and $g(x) = \sqrt{x^2-1}$ must be real numbers and $g(x)$ cannot be 0. We already know from part (a) that $f(x)$ and $g(x)$ are real whenever $x \in (\infty, -1] \cup [1, 3]$, and also that
   $$\sqrt{x^2-1} = 0 \iff x^2 - 1 = 0 \iff (x+1)(x-1) = 0 \iff x = -1 \text{ and } x = 1.$$
   Therefore we cannot have -1 and 1 in the domain, and the largest possible domain is
   $$D\left(fg\right) = (\infty, -1) \cup (1, 3].$$

2. (5 pts each) Let f and g be functions of real numbers such that f can be composed with g. Suppose g is an odd function and let $h = f \circ g$.

   (a) Is h always an odd function? If you think it is, prove that it is; if you think it is not, find a counterexample.

   No, h is not always odd. Here is a counterexample. Let $f, g : \mathbb{R} \to \mathbb{R}$ be the following functions
   $$\begin{aligned} f(x) &= x^2 \\ g(x) &= x \\ h(x) &= f \circ g(x) = f(g(x)) = x^2 \end{aligned}$$

Note that $g(-x) = -x = -g(x)$ for every real number x, so g is indeed an odd function. But h is not odd because

$$h(-x) = (-x)^2 = x^2 \neq -x^2 = -h(x)$$

in general. For example, $h(-1) = 1$ and $-h(1) = -1$ are not equal.

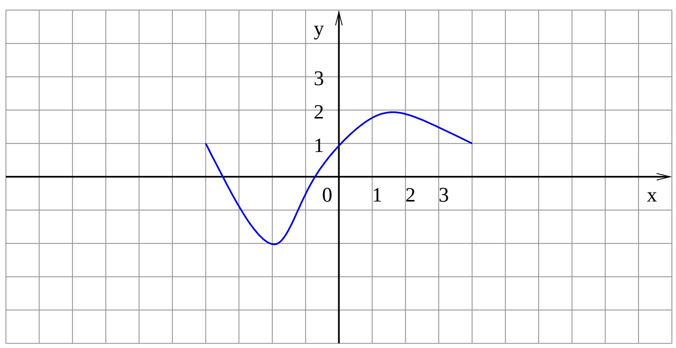
(b) What if f is odd? Is h always an odd function then? If you think it is, prove that it is; if you think it is not, find a counterexample.

Yes, h is always an odd function in this case. Let f and g be odd functions. Then

$$\begin{aligned} h(-x) &= f(g(-x)) & \\ &= f(-g(x)) & \text{because } g \text{ is odd} \\ &= -f(g(x)) & \text{because } f \text{ is odd} \\ &= -h(x) & \end{aligned}$$

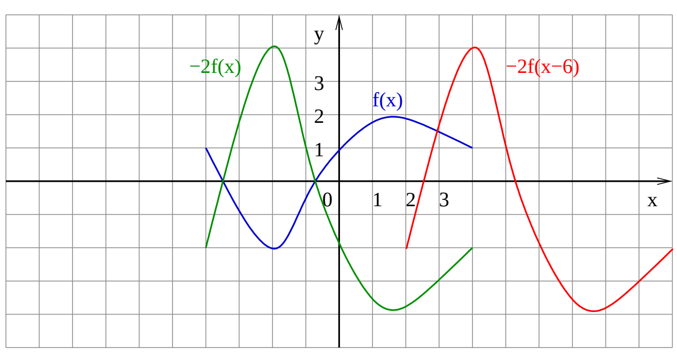
for every x.

3. (5 pts each) The graph of a function $f(x)$ is given below.



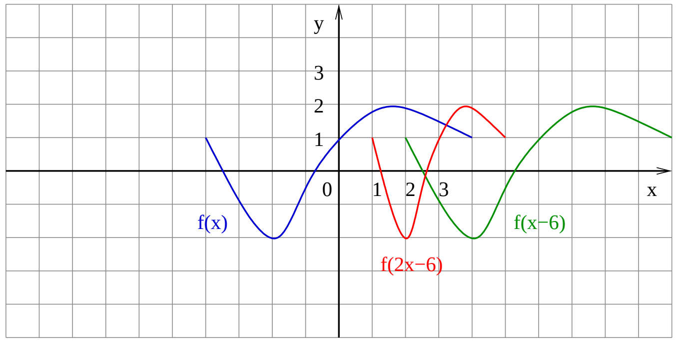
Sketch the graphs of the following functions. Make sure you explain your work.

(a) $g(x) = -2f(x-6)$



Subtracting 6 from the input variable shifts the graph right by 6 units, multiplying the output by -2 stretches the graph vertically by a factor of 2 and reflects it across the x-axis. The order of these transformations does not matter.

(b) $h(x) = f(2x-6)$



Subtracting 6 from the input variable shifts the graph right by 6 units, multiplying the input by 2 compresses the graph horizontally by a factor of 2. The order does matter in this case: the shift comes before the compression.

4. (a) (4 pts) Let f be a function of real numbers and $a \in \mathbb{R}$ such that in some neighborhood of a, $f(x)$ has a value for x except possibly at $x = a$. State the informal definition of

$$\lim_{x \to a} f(x) = L,$$

where L is a real number.

This is Definition 1 in Section 1.3. You can see the textbook's version there. This is how we said it in class:

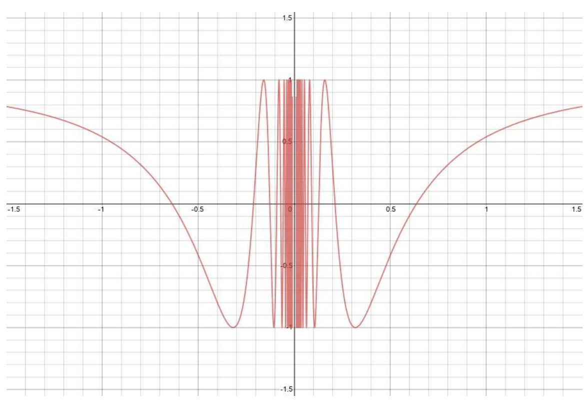
$$\lim_{x \to a} f(x) = L$$

means that the value of $f(x)$ can be made arbitrarily close (as close as you want) to L if x is made sufficiently close to a but not equal to a.

(b) (6 pts) The graph of the function

$$f(x) = \cos\left(\frac{1}{x}\right)$$

is shown below.



Explain why

$$\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$$

does not exist.

Notice that the graph is very similar to the one in Example 5 in Section 1.3, and it suggests that the reason the limit does not exist is that the value of $f(x)$ oscillates up and down between -1 and 1 no matter how close x is to 0. In fact, we can easily see this by considering what happens if $x = \frac{1}{n\pi}$ where n is an integer. First, notice that no matter how close to 0 we look, x will have such values since if n is large or negative enough, $\frac{1}{n\pi}$ can be as close to 0 as we want. Now,

$$\cos\left(\frac{1}{x}\right) = \cos\left(\frac{1}{\frac{1}{n\pi}}\right) = \cos(n\pi),$$

which is 1 if n is even and -1 if n is odd. So no matter how close x is to 0, it can have a value for which $f(x) = 1$ or a value for which $f(x) = -1$. Or it can be any number between -1 and 1 for that matter. In any case, the value of $f(x)$ does not get arbitrarily close to an particular number, but continues to oscillate between -1 and 1.

5. (5 pts each) **Extra credit problem.**

   (a) Let f be an even function and let a be a real number such that

   $$\lim_{x\to a^-} f(x) = L.$$

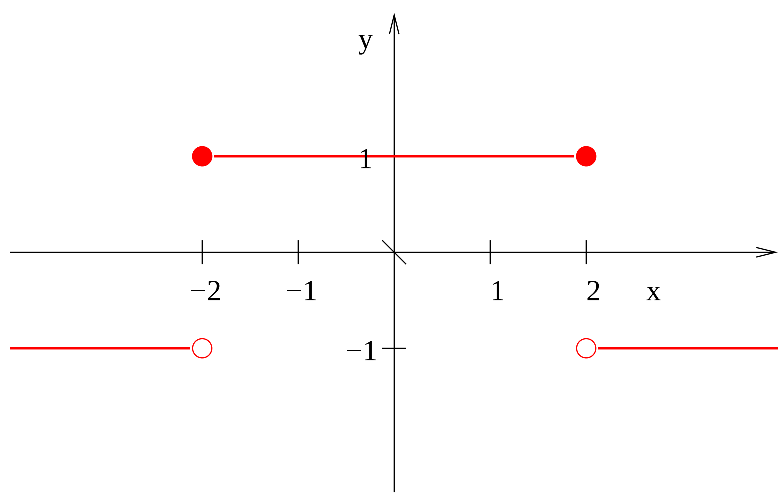
   Does

   $$\lim_{x\to -a^-} f(x) = L$$

   have to be true?

   This does not have to be true. Here is a counterexample.

   $$f(x) = \begin{cases} 1 & \text{if } -2 \le x \le 2 \\ -1 & \text{if } x < -2 \text{ or } x > 2 \end{cases}$$



   First, let us check that f is an even function. Let x be any real number. If $x \in [-2, 2]$ then $-x \in [-2, 2]$. So $f(-x) = 1 = f(x)$. If $x < -2$ then $-x > 2$, and so $f(-x) = -1 = f(x)$. Finally, if $x > 2$ then $-x < -2$, and so $f(-x) = -1 = f(x)$. In all cases, $f(-x) = f(x)$, which shows that f is even.
   Now

   $$\lim_{x\to 2^-} f(x) = 1$$

   because for any number $x < 2$ close to 2, $f(x) = 1$. But

   $$\lim_{x\to -2^-} f(x) = -1$$

   because for any number $x < -2$ close to -2, $f(x) = -1$.

(b) Now, let g be an odd function and let a be a real number such that

$$\lim_{x\to a^+} g(x) = L.$$

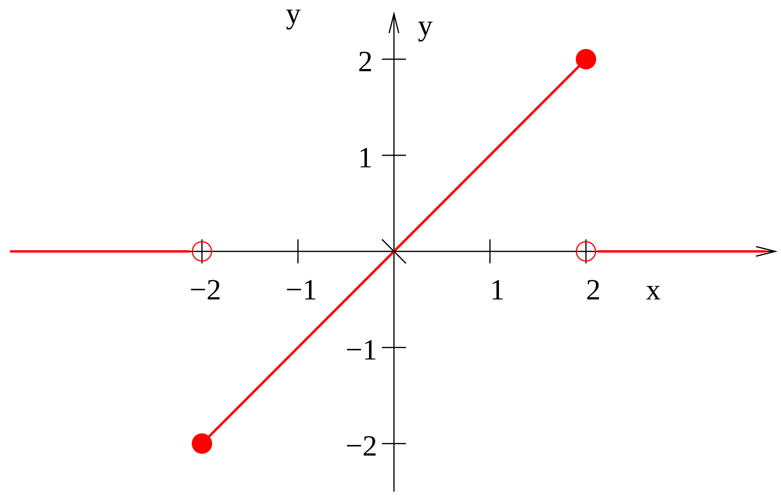
Does

$$\lim_{x\to -a^+} g(x) = -L$$

have to be true?

This does not have to be true either. Here is a counterexample.

$$g(x) = \begin{cases} x & \text{if } -2 \le x \le 2 \\ 0 & \text{if } x < -2 \text{ or } x > 2 \end{cases}$$



First, let us check that g is an odd function. Let x be any real number. If $x \in [-2, 2]$ then $-x \in [-2, 2]$. So $g(-x) = -x$ and $g(x) = x$, and hence $g(-x) = -g(x)$. If $x < -2$ then $-x > 2$, and so $g(-x) = 0$ and $g(x) = 0$, and hence $g(-x) = -g(x)$. Finally, if $x > 2$ then $-x < -2$, and so $g(-x) = 0$ and $g(x) = 0$, and hence $g(-x) = -g(x)$. In all cases, $g(-x) = -g(x)$, which shows that g is odd.
Then

$$\lim_{x\to 2^+} g(x) = 0$$

because for any number $x > 2$ close to 2, $g(x) = 0$. But

$$\lim_{x\to -2^+} g(x) = -2$$

because for any number $x > -2$ close to -2, $g(x) = x$, and as as x approaches -2 from the right, $g(x) = x$ also approaches -2.