MCS 118 Final Exam Solutions

1. (5 pts each)
   (a) Determine whether the function $f : \mathbb{R} \to \mathbb{R}$ given by
   $$f(x) = \frac{x^2}{x^4+1}$$
   is even, odd, or neither. Why?

   This function is even since
   $$f(-x) = \frac{(-x)^2}{(-x)^4+1} = \frac{x^2}{x^4+1} = f(x)$$
   for every real number x.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. If f is an even function and g is an odd function, what can you say about the symmetry of fg? Is fg even or odd or both or neither in this case? Do not forget to justify your answer.

   In this case, fg is odd because
   $$\begin{aligned}(fg)(-x) &= f(-x)g(-x) \\ &= f(x)[-g(x)] && \text{by } f(-x) = f(x) \text{ and } g(-x) = -g(x) \\ &= f(x)g(x) \\ &= -(fg)(x)\end{aligned}$$
   for all x.

2. (10 pts) Let
   $$f(x) = \frac{x}{1+x} \quad \text{and} \quad g(x) = \sin(2x).$$
   Find $f \circ g$ and the largest possible subset of the real numbers that could be the domain of $f \circ g$.

   $$f \circ g(x) = f(g(x)) = \frac{\sin(2x)}{1+\sin(2x)}.$$
   Since $\sin(2x)$ is a real number for any $x \in \mathbb{R}$, $f \circ g(x)$ has a legitimate value for any rumber x as long as $1 + \sin(2x) \neq 0$. Now,
   $$1 + \sin(2x) = 0 \iff \sin(2x) = -1.$$
   The value of the sine function is -1 when its input is $3\pi/2$ plus any integer multiple of 2π (think about the unit circle of the graph of the sine function). So
   $$2x = \frac{3\pi}{2} + 2k\pi \iff x = \frac{3\pi}{4} + k\pi$$
   where k is any integer. Hence
   $$D(f \circ g) = \{x \in \mathbb{R} \mid x \neq \frac{3\pi}{4} + k\pi \text{ for any } k \in \mathbb{Z}\}.$$

3. (10 pts) Evaluate

$$\lim_{u\to -2} \sqrt{u^4+3u+6}$$

and justify each step by indicating the appropriate Limit Law(s) you used.

$$\begin{aligned}
\lim_{u\to -2} \sqrt{u^4+3u+6} &= \sqrt{\lim_{u\to -2}(u^4+3u+6)} && \text{by LL11}\\
&= \sqrt{\lim_{u\to -2} u^4 + \lim_{u\to -2}(3u) + \lim_{u\to -2} 6} && \text{by LL1}\\
&= \sqrt{(-2)^4 + 3\lim_{u\to -2} u + 6} && \text{by LL9, LL3, and LL7}\\
&= \sqrt{(-2)^4 + 3(-2) + 6} && \text{by LL8}\\
&= \sqrt{16}\\
&= 4
\end{aligned}$$

Note that u^4+3u+6 is a polynomial, so after using LL11, we could have evaluated $\lim_{u\to -2}(u^4+3u+6)$ by direct substitution too. But the problem asked for using the Limit Laws, so that it what I did.

4. (a) (4 pts) Let f be a function whose inputs and outputs are real numbers. Define what it means for f to be a decreasing function.

   Such a function f is decreasing if $f(x_1) > f(x_2)$ for all real numbers $x_1 < x_2$ in the domain of f.

   (b) (6 pts) Give an example of a nonlinear function that is decreasing. Justify your example.

   There are of course many such examples. One possibility is $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = 1/x$. This is clearly not a linear function because it is not of the form $f(x) = mx + b$. Suppose $x_1 < x_2$ are both positive real numbers (since the domain of f is $\mathbb{R}^+$). Then

$$x_1 < x_2 \implies \frac{x_1}{x_2} < 1 \implies \frac{1}{x_2} < \frac{1}{x_1} \implies f(x_2) < f(x_1).$$

5. (10 pts) Use the formal definition of the limit to prove that

$$\lim_{x\to -3}(4x+10) = -2.$$

We need to show that for every $\epsilon > 0$, there is a corresponding $\delta > 0$ such that if

$$0 < |x-(-3)| < \delta$$

then

$$|(4x+10)-(-2)| < \epsilon.$$

So we want

$$|(4x+10)-(-2)| < \epsilon.$$

Notice that

$$|(4x+10)-(-2)| = |4x+10+2)| = |4x+12| = |4(x+3)| = |4||x+3| = 4|x+3|.$$

This suggests that we should make $|x+3| < \epsilon/4$.

So let $\delta = \epsilon/4$. Suppose

$$0 < |x - (-3)| < \delta = \frac{\epsilon}{4}.$$

Then

$$|(4x + 10) - (-2)| = 4|x + 3| = 4|x - (-3)| < 4\,\frac{\epsilon}{4} = \epsilon,$$

which is what we wanted to show.

6. Let c be a real number and define the function

$$f(x) = \begin{cases} x^2 - cx + 5 & \text{if } x < -2 \\ \frac{c}{x+1} & \text{if } x \geq -2. \end{cases}$$

where $x \neq -1$.

(a) (4 pts) Find

$$\lim_{x \to -2^+} f(x)$$

and fully justify your answer.

Since $x > -2$, $f(x) = \frac{c}{x+1}$. This is a rational function and $x = -2$ does not make its denominator 0, so the easiest way to find its limit as $x \to -2^+$ is by direct substitution:

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \frac{c}{x+1} = \frac{c}{-2+1} = \frac{c}{-1} = -c.$$

(b) (3 pts) Find

$$\lim_{x \to -2^-} f(x)$$

and fully justify your answer.

Since $x < -2$, $f(x) = x^2 - cx + 5$. So

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (x^2 - cx + 5) = (-2)^2 - c(-2) + 5 = 9 + 2c$$

by direct substitution into the polynomial $x^2 - cx + 5$.

(c) (3 pts) For what value of c does

$$\lim_{x \to -2} f(x)$$

exist? Why?

By one of the theorems we learned in class (Theorem 2 in Section 1.4), the two-sided limit exists if and only if the one-sided limits both exist and are equal. We already know from parts (a) and (b) that the one-sided limits exist. They are equal when

$$-c = 9 + 2c \iff -3c = 9 \iff c = -3.$$

So $\lim_{x \to -2} f(x)$ exists if and only if $c = -3$.

7. (10 pts) **Extra credit problem.** We have learned that Limit Law 4 says if

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

both exist then

$$\lim_{x \to a} [f(x)g(x)]$$

also exists, and in fact,

$$\lim_{x \to a} [f(x)g(x)] = \left[\lim_{x \to a} f(x)\right] \left[\lim_{x \to a} g(x)\right].$$

Find an example of two functions f and g such that neither

$$\lim_{x\to a} f(x) \quad \text{nor} \quad \lim_{x\to a} g(x)$$

exists, yet

$$\lim_{x\to a}[f(x)g(x)]$$

does exist. Justify your example.

Here is one such example. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \frac{|x|}{x}$$

and let g be the same function. We have seen in class that

$$\lim_{x\to 0} f(x)$$

does not exist because the one-sided limits

$$\lim_{x\to 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x\to 0^-} f(x) = -1$$

are not equal. For the same reason,

$$\lim_{x\to 0} g(x)$$

does not exist either. Notice

$$(fg)(x) = f(x)g(x) = \frac{|x|}{x}\frac{|x|}{x} = \frac{|x|^2}{x^2} = \frac{x^2}{x^2} = 1$$

whenever $x \neq 0$. But it does not matter what happens when $x = 0$ as far as the limit

$$\lim_{x\to 0}[f(x)g(x)]$$

is concerned. Therefore

$$\lim_{x\to 0}[f(x)g(x)] = \lim_{x\to 0} 1 = 1$$

by Limit Law 7.