Notes for Section 1.1

Calculus studies functions of real numbers. Therefore we start by looking at what a function is. You have certainly come across functions before in your math classes. You may feel that you have a good understanding of what a function is. Perhaps you do, and if you do, great. But my experience is that many students taking calculus for the first time have serious misconceptions of what a function. So let me start by pointing out what a function is not: it is not a formula or expression into which you can substitute numbers. Expressions are in fact often used to describe functions and can be an essential ingredient in constructing a function. But an expression by itself is not a function.

Let me take a moment to talk about expressions versus equations:

$$3x^2 + 2x - 1$$

is an expression, whereas

$$3x^2 + 2x - 1 = 5$$

is an equation. Can you see how they differ? If it does not have an equal sign in it, it is not an equation. It is easy to tell the difference. Don't confuse them.

So what is a function? We will use the following informal definition:

Definition 1. A *function* f from a set S to a set T is a rule which assigns to every element x in S one and only one element $f(x)$ in T. The set S is called the *domain* of f and the set T is called the *codomain*.

I will give you a couple of examples shortly, but let me first introduce some shorthand notation. Instead of writing "x is an element of the set S," we will often use the shorthand $x \in S$. The $\in$ symbol means "element of." Instead of writing "f is a function from the set S to the set T," we will typically write $f : S \to T$.

So here are some examples.

Example 1. *Let S be the set of all undergraduate students enrolled at Gustavus in fall 2020 and let T be the set of all 6-digit or 7-digit positive integers. Let $f : S \to T$ be the rule $f(x) =$ x's student ID number. Then f is a function because f assigns to each element of S, i.e. each student enrolled at Gustavus one and only one element of T because every student enrolled at Gustavus has exactly one student ID number and that is a 6-digit or a 7-digit positive integer.*

Example 2. *Let S be the set of all undergraduate students enrolled at Gustavus in fall 2020 and let T be the set of all 6-digit positive integers. Let $f : S \to T$ be the rule $f(x) = x$'s student ID number. This is very similar to the previous example, but notice the difference. This f is not a function because some students in S certainly have 7-digit student ID numbers, and for such a student x, f would not be able to assign an element in T.*

Example 3. *Let S be the set of all 6-digit or 7-digit positive integers and let T be the set of all undergraduate students enrolled at Gustavus in fall 2020. Let $f : S \to T$ be the rule $f(x) =$ the student whose ID number is x. Then f is not a function because there certainly are some 6-digit and 7-digit numbers that are not the ID numbers of students currently enrolled at Gustavus, and if x is such a number, f does not assign any element of T to x. How do I know? Think about how many 6-digit and 7-digit numbers there are and how many students are enrolled at the college this semester.*

Example 4. *Let S be the set of all undergraduate students enrolled at Gustavus in fall 2020 and let T be the set of all pets. Let $f : S \to T$ be the rule $f(x) = x$'s pet. Then f is a not a function because some student x in S could have more than one pet or no pet at all. For f to be a function, each student x at Gustavus would have to have exactly one pet.*

Example 5. *Let S be the set of all undergraduate students living on campus at Gustavus in fall 2020 and let T be the set of all dorm rooms at GAC. Let $f : S \to T$ be the rule $f(x) = x$'s dorm room. Then f is a function because every student x in S lives in one and only one room. By specifying that every element of S is a student living on campus, we made sure that f can always assign a dorm room in T to every x. And we can be certain that no student lives in more than one room. Now, what if two students share the same room? Say x_1 and x_2 are two students who live on campus and are roommates. Then $f(x_1) = f(x_2)$. Which is fine, because f is allowed to assign the same element in T to two different elements in S. What it is not allowed to do is assign more than one element in T to the same element in S or assign no element in T to an element in S.*

Example 6. *Let $f(x) = \sqrt{x}$. Then f is not a function. We cannot even decide whether f assigns to every element in its domain one and only one element of its codomain because we do not know what the domain and the codomain are.*

Example 7. *Let $S = \mathbb{R}^{\geq 0}$ be the set of all nonnegative real numbers and let $T = \mathbb{R}$ be the set of all real numbers. Let $f : S \to T$ be the rule $f(x) = \sqrt{x}$. Then f is a function because every nonnegative real number has one and only one square root, so f does indeed assign to every element in its domain one and only one element of its codomain.*

What you are supposed to glean from the above examples is that the domain and the codomain are essential ingredients of a function, as is a precise enough description of the rule by which f assigns elements of its codomain to elements of its domain. What the domain and the codomain are can make or break a function.

Before I charge on, let me take some time to point out the importance of definitions in mathematics. We have just come across our first definition in this course, the definition of a function. In your textbook, it is framed by a blue box. That is to alert you that this is really important. Mathematicians are sticklers for precision and that includes precise language too. Definitions allow us to speak the same language. You must develop a healthy respect for them. Learn them well. This does not mean you need to memorize them word for word. It is possible to state the same definition in different words. For example, I used somewhat different wording to define a function from the textbook. I said "one and only one" where to book says "exactly one." That is fine because they mean the same thing. I also used S and T for the domain and the codomain while the book uses D and E. That is also fine because what really matters is the role the domain and the codomain play in the definition, not the specific letters used to denote them. You may even have noticed that the book never tells you that the name of T or E is the codomain. The important thing for learning definitions is to understand what each part of the definition means and why it is important. This is why we study examples.

I said above that the definition of function we would use in this course is an informal one. So what is informal about it? It is the "rule." We never explained exactly what a rule is. We are leaving it up to our everyday intuition to understand what a rule means. There is a formal definition of function, but it is fairly technical and it most likely would not help you develop the understanding of functions you need in a calculus course. If you stay tuned and take upper level math courses, you will see it.

Here is some useful terminology, which our textbook uses and we will too. Let $f : S \to T$, that is f is a function from the set S to the set T. When we write $f(x)$, the element x of S is the *input value* or simply *input* to f. If you want to think about x as having varying values, you can refer to it as the *input variable* or *independent variable*. Similarly, $f(x)$ is the *output value*, or simply *output* or *value* of f. If we write $y = f(x)$, we can refer to y as the *output variable* or *dependent variable* of f. Figure 2 in the textbook illustrates where the input/output terminology comes from. The dependent/independent terminology has to do with the fact that in $y = f(x)$ the value of y depends on the value of x, while the value of x can be chosen freely or independently to be any element of S. One more useful concept is the *range* of a function $f : S \to T$. The range of f is the set

$$R(f) = \{f(x) \mid x \in S\},$$

that is the set of all output values of f as the input x takes on all possible values in the domain. So what is the difference between the codomain and the range? Every element of the range $R(f)$ is an actual output value of f. Every output value of f is an element of $R(f)$ and every element of $R(f)$ is an output value of f. The codomain must include all possible output values of f but not every element of the codomain needs to be an output value of f. In other words, the range $R(f)$ is always a subset of the codomain T but it may be a proper subset, that is not all of T. E.g. in Example 7 above, the codomain of f is the set of all real numbers $\mathbb{R}$, but the range is only the set of all nonnegative real numbers $\mathbb{R}^{\geq 0}$ because the square root of a number is never negative, while every positive real number is indeed the square root of a real number, and so is 0.

Spend some time studying the arrow diagram in Figure 3. This is a very useful way to visualize what a function does. I will show you in class how you can use arrow diagrams to make good sense of the "one and only one" part of the definition of a function. It also illustrates why I (and many other people) like S and T to denote the domain and the codomain: S stands for the sources and T stands for the targets of the arrows in the diagram.

Definition 2. The *graph of a function* $f : S \to T$ is the set of *ordered pairs*

$$G(f) = \{\big(x, f(x)\big) \mid x \in S\}.$$

Example 8. *Let* $S = \{cat, dog, chicken, snail\}$ *and* $T = \mathbb{Z}$*, the set of all integers. Let* $f : S \to T$ *be defined by* $f(x) =$ *the typical number of legs* x *has. Then*

$$G(f) = \Big\{(cat, 4), (dog, 4), (chicken, 2), (snail, 0)\Big\}.$$

Example 9. *Let* f *be the square root function in Example 7 above. Then*

$$G(f) = \{(x, \sqrt{x}) \mid x \in \mathbb{R}^{\geq 0}\}.$$

and contains ordered pairs such as $(4, 2)$*,* $(9, 3)$*,* $(2, \sqrt{2})$*,* $(0, 0)$*, etc. You can visualize this as a set of points in a Cartesian coordinate system in the plane, using the first coordinate of each order pair as the horizontal coordinate and the second coordinate as the vertical coordinate. I will leave drawing the graph as an exercise to you.*

Notice that the graph in the first example cannot easily be visualized in a coordinate system in the plane while the graph in the second example can. Calculus is the study of functions whose inputs and outputs are both real numbers (let me refer to these from now on as *functions of real numbers*), so most of the functions we will study in this course will have a graph that can visualized as a subset of the plane. This is a very useful tool for studying the behavior of a function and we will use it often. But functions whose domain or codomain or both are not subsets of the real numbers still have graphs, they are just not subsets of the plane.

The book talks about four common ways of representing the rule of a function:

- **By words:** just like I did in the first five examples of functions above,
- **By formula:** just like I did in the last two examples of functions above,
- **By a table of values:** there are two examples of this on the left margin on pp. 3 and 4 in the book,
- **By graph:** see Example 2 and Figure 9 in the book.

These are certainly not the only way to represent a function, but they may be the four most common ways. Familiarize yourself with each. Calculus is the study of functions whose inputs and outputs are both real numbers. Thus we will most often use a formula or expression to describe a function in this class. But do not forget that a function is not a formula.

Here is an important observation about representing functions of real numbers by graphs: a curve in the plane or any subset of the plane) is the graph of the graph of a function of real numbers if and only if no vertical line intersects it more than once. Think carefully about why this is. We represent the output values on the vertical axis. If a vertical line intersects a curve more than once, what does that tell you about the number of output values that would have to assigned to the same input value?

Sometimes we can represent a function by dividing the domain into several subsets and giving different descriptions of the rule it uses to assign output values to input values on each subset. Here is an important example, using two algebraic formulas on different parts of the domain:

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

This just means that when you evaluate the function at an input value, you first look up which condition the input value satisfies, then you use the corresponding description. E.g. for the above function, $f(4) = 4$, $f(-3) = -(-3)$, and $f(0) = 0$. In fact, you have likely recognized already that the function I have just defined is the well-known absolute value function. Let me repeat the definition because we will use the absolute value function repeatedly in this course, so I want you to remember well what it does:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

Such functions are called *piecewise defined functions*. Make sure you understand the notation and know how to use it to describe your own piecewise defined functions. Here is another example of a function with an absolute value in it and how to express it as a piecewise defined function. Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = |2x - 3|$. First, note that $|2x - 3|$ is either $2x - 3$ or $-(2x - 3)$ depending on the sign of $2x - 3$. If $2x - 3 \geq 0$ then $|2x - 3| = 2x - 3$, and if $2x - 3 < 0$ then $|2x - 3| = -(2x - 3) = 3 - 2x$. Now,

$$\begin{aligned} 2x - 3 &\geq 0 \\ 2x &\geq 3 \\ x &\geq \frac{3}{2}. \end{aligned}$$

So whenever $x \geq 3/2$, $g(x) = 2x - 3$. Similarly,

$$\begin{aligned} 2x - 3 &< 0 \\ 2x &< 3 \\ x &< \frac{3}{2}. \end{aligned}$$

So whenever $x < 3/2$, $g(x) = 3 - 2x$. Therefore

$$g(x) = \begin{cases} 2x - 3 & \text{if } x \geq \frac{3}{2} \\ 3 - 2x & \text{if } x < \frac{3}{2} \end{cases}.$$

Definition 3. A function of real numbers f is called *even* if $f(-x) = f(x)$ for every x in its domain and is called *odd* if $f(-x) = -f(x)$ for every x in the domain.

For example, the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is even because it is true that

$$f(-x) = (-x)^2 = (-x)(-x) = x \cdot x = x^2 = f(x)$$

for every $x \in \mathbb{R}$. On the other hand, the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^3$ is odd because

$$g(-x) = (-x)^3 = (-x)(-x)(-x) = -x \cdot x \cdot x = -x^3 = -g(x)$$

for every $x \in \mathbb{R}$. It is really important to understand that for a function f to be even, $f(-x) = f(x)$ must be true for every x in the domain. You cannot check if a function is even by looking at a particular element in the domain. For example, $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = x^3 - 3x^2 - x$ does satisfy $h(-x) = h(x)$ when $x = 1$, but is not an even function because $h(-x) = h(x)$ is not true for other values of x. The same goes for odd functions. To decide if a function f is even or odd or neither, you would have to look at the value of $f(-x)$ at a generic number x and see if it matches $f(x)$ or $-f(x)$ or neither of them. For example, it follows from the definition of the sine function on the unit circle that

$$\sin(-x) = -\sin(x)$$

for any angle x. So $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin(x)$ is an odd function. It is not an even function because for example $\sin(-\pi/2) = -1 \neq 1 = \sin(\pi/2)$. You can show that a function is not even (or not odd) by looking at a specific number x and verifying that $f(-x) \neq f(x)$ (or $f(-x) \neq -f(x)$), but you cannot show that it is even or odd by looking at one specific value of x. Another example of an even function would be the absolute value function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$. It is easy enough to see that

$$f(-x) = |-x| = |x| = f(x)$$

for any real number x, whether x is positive or negative or 0. By the way, $|-x| \neq x$ in general. Try $x = -2$

Evenness and oddness are called symmetries of functions. You can understand why if you think about the graph. The graph of an even function f is symmetric about the y-axis. This is because it contains both the point $(x, f(x))$ and the point $(-x, f(x))$ for every number x in its domain, and those two points are mirror images of each other across the y-axis. Similarly, the graph of an odd function g is symmetric about the origin because it contains both the point $(x, f(x))$ and the point $(-x, -f(x))$ for every number x in its domain, and those two points are mirror images of each other across the origin.

A function does not need to be either even or odd. It can be neither or even both. It is easy enough to come up with a function whose graph is neither symmetric about the y-axis nor about the origin. Can you think of a function whose graph has both symmetries?

Definition 4. A function of real numbers f is *increasing* on an interval I if it is true for any $x_1 < x_2$ in I that $f(x_1) < f(x_2)$. Similarly, f is *decreasing* on an interval I if it is true for any $x_1 < x_2$ in I that $f(x_1) > f(x_2)$.

These definitions are consistent with your intuition, which likely tells you that the graph of an increasing function should be going up while the graph of a decreasing functions should be going down. But saying that a function is increasing if its graph is going up would not be clear and precise enough to be acceptable as a definition. Note that in some other books, such functions are called strictly increasing and decreasing, while just increasing and decreasing allow the equality of $f(x_1)$ and $f(x_2)$ as well.

For example, the absolute value function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$ is increasing on $[0, \infty)$ because it is true that for any two numbers $x_1, x_2 \in [0, \infty)$ such that $x_1 < x_2$,

$$f(x_1) = |x_1| = x_1 < x_2 = |x_2| = f(x_2).$$

But f is decreasing on $(-\infty, 0)$ because for any two numbers $x_1, x_2 \in (-\infty, 0)$ such that $x_1 < x_2$,

$$f(x_1) = |x_1| = -x_1 > -x_2 = |x_2| = f(x_2).$$

Think about a few more examples. On what intervals is the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ increasing? On what intervals is it decreasing? What about $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$?

There is one more thing in this section I have not talked about. Example 3 asks to find the domains of two functions

$$f(x) = \sqrt{x+2} \quad \text{and} \quad g(x) = \frac{1}{x^2 - x}.$$

The book really gets this wrong. The two expressions are just algebraic expressions, nothing more. For them to be functions, they would have to have clearly specified domains and codomains. But if they did have clearly specified domains, then it would not make sense to ask to find those. In fact, the question does not make sense as stated. A function always has to come equipped with a domain, and what the domain is cannot be determined from a formula. Nevertheless, the kind of calculation and thinking the example illustrates are quite useful in practice. The answers given in the textbook are correct; the question is not. So let us salvage this example by rephrasing the question in a way that makes sense: what is the largest subset S of the real numbers that could be domain of the function $f : S \to \mathbb{R}$ given by $f(x) = \sqrt{x+2}$? And similarly for g. You will be asked a few similar questions on the homework. Every time you see one of these questions, think about how the question would sound if it were asked correctly.