Notes for Section 1.2

An important reason that people care about functions and want to study them is that they are the building blocks of mathematical models in the physical and social sciences. This section is a reminder of the basic kinds of functions of real numbers you have come across in your math classes and the ways you can combine them to build other functions.

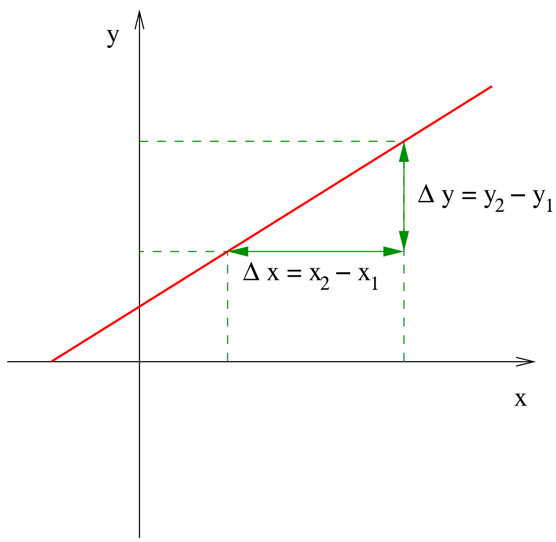
A *linear function* from real numbers to real numbers is a function of the form $f(x) = mx + b$, where m and b are constants. They are called linear functions because their graphs are straight lines. The reason that the graphs are straight lines is that the function has a constant *slope*. The slope between a point (x_1, y_1) and a point (x_2, y_2) is the ratio of the difference in the y-coordinates and the difference in the x-coordinates:

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

We are, of course, assuming that $x_1 \neq x_2$, otherwise the denominator is 0 and the ratio is not defined. Notice that it does not matter which of the two points you consider the first point and which is the second point because

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{-(y_2 - y_1)}{-(x_2 - x_1)} = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope, as the name may suggest, has to do with how steeply the line that passes through the two points goes up (if the slope is positive) or down (if the slope is negative). If we denote the difference $y_2 - y_1$ by Δy and the difference $x_2 - x_1$ by Δx, then the slope is also $\frac{\Delta y}{\Delta x}$. It is customary to refer to Δy as the *rise* and to Δx as the *run*. Hence the slope is also the rise divided by the run, or "rise over run."



What makes a linear function special is that its slope is the same between any two (distinct) points on its graph. Let $f(x) = mx + b$ and let (x_1, y_1) and (x_2, y_2) be two (distinct) points on its graph. Then $y_1 = f(x_1)$ and $y_2 = f(x_2)$. The slope between them is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = \frac{mx_2 + b - mx_1 - b}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} = m.$$

So m is always the slope regardless of the values of x_1 and x_2.

The value of b is the *y-intercept*. The graph of $f(x) = mx + b$ crosses the y-axis at $y = b$ or at the point $(0, b)$ if you prefer to specify both coordinates. Finding the equation of a line that passes through two given points or one whose slope is given along with a point on the line are common problems and you should know how to do them.

One more thing worth mentioning is that the slope of a vertical line is not defined and its equation cannot be written in the form $y = mx + b$. But a vertical line is not the graph of a function anyway because it fails the vertical line test in an obvious way. Vertical lines do have equations, of a very simple form $x = c$ where the point $(c, 0)$ is where the line crosses the x-axis.

Power functions are of the form $f(x) = x^n$ where n is a constant number, or more generally, $f(x) = kx^n$ where k and n are constants. The value of n is called the *degree* of the power function and determines the shape of the graph. If n is a positive even integer, the graph is similar to the graph of $f(x) = x^2$, although it is not actually a parabola, unless $n = 2$. The larger the value of n, the steeper the graph as x increases in the positive direction or decreases toward more negative values. If n is a positive odd integer, the graph is similar to the graph of $f(x) = x^3$. The larger the value of n, the steeper the graph. You should be familiar with the following algebraic properties of powers (or exponents, depending on how you look at them) and understand why they hold.

$$\begin{aligned}
x^m x^n &= x^{m+n} \\
\frac{x^m}{x^n} &= x^{m-n} \\
x^{-n} &= \frac{1}{x^n} \\
x^0 &= 1 && \text{for any } x \in \mathbb{R} \text{ except } x = 0 \\
\left(x^m\right)^n &= x^{mn} \\
(xy)^n &= x^n y^n
\end{aligned}$$

Note that 0^0 is undefined. It is easy to understand why these work if m and n are positive integers, but they also work when m and n are any real numbers.

You should also be familiar with fractional exponents. First,

$$x^{1/n} = \sqrt[n]{x}$$

since

$$\left(x^{1/n}\right)^n = x^{(1/n)n} = x^1 = x,$$

and so $x^{1/n}$ must be a number whose n-th power is x. The n-th root of x is exactly such a number. Note that even roots such as $\sqrt{\ }$, $\sqrt[4]{\ }$, $\sqrt[6]{\ }$, etc, do not like negative numbers under the root because any even power or any real number is nonnegative. So there is no real number that could be the value of $\sqrt[n]{x}$ if n is even and $x < 0$. Odd roots on the other hand are not picky about negative numbers. E.g. $\sqrt[3]{-8} = -2$ as $(-2)^3 = -8$.

If there is a fraction in the exponent whose numerator is not 1, it can be interpreted in two ways:

$$x^{m/n} = x^{(1/n)m} = \left(x^{1/n}\right)^m = \left(\sqrt[n]{x}\right)^m$$

or

$$x^{m/n} = x^{m(1/n)} = \left(x^m\right)^{1/n} = \sqrt[n]{x^m}.$$

Both are correct and in general,

$$\left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}.$$

There is a bit of ambiguity when m/n is not reduced to lowest terms. E.g.

$$(-5)^{6/4} = \left(\sqrt[4]{-5}\right)^6$$

has no real value because of the negative number in the 4th root, whereas

$$(-5)^{6/4} = \sqrt[4]{(-5)^6} = \sqrt[4]{15625} \approx 11.1803.$$

The convention in this case it that the fraction should first be reduced to lowest terms. So in the last example, $6/4 = 3/2$ and now

$$(-5)^{3/2} = \left(\sqrt{-5}\right)^3$$

has no real value because of the negative number in the square root. In fact,

$$(-5)^{3/2} = \sqrt{(-5)^3} = \sqrt{-125}$$

has no real value either for the same reason. The ambiguity is gone and this is the correct interpretation of $(-5)^{3/2}$.

You should be familiar with graphs of power functions (including root functions) and understand how the exponent affects the shape of the graph of $f(x) = x^n$. Play with Desmos if you need to.

Polynomials are sums of power functions with nonnegative integer exponents, such as $p(x) = 3x^4 - 2x^3 + 7x + 5$. In general, a polynomial is of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x^1 + a_0$$

where n is a nonnegative integer. It is important to understand that polynomials cannot have powers of x with negative or noninteger exponents. The $a_0, a_1, \ldots, a_n$ are called the *coefficients* of the polynomial. The term $a_n x^n$ with the highest exponent such that $a_n \neq 0$ is called the *leading term* of the polynomial and its coefficient a_n is the *leading coefficient.* The *degree* of the polynomial is the degree of the leading term. The term a_0 is the *constant term.* If you are missing how that term is a power function, you can think of it as a_0x^0 since $x^0 = 1$ in most cases. Except of course that 0^0 is not 1, but we will not split hairs over this.

Example:

$$p(x) = 3x^4 - 2x^3 + 7x + 5$$

is a polynomial of degree 4, whose leading term is $3x^4$, leading coefficient is 3, and constant term is 5. It is worth noting that linear functions are polynomials of degree 1 (of the slope is nonzero) or 0 (if the slope is 0). In fact, a linear function whose slope is 0 is just a constant function $f(x) = c$. The degree 0 of such a constant function is 0 because you can always write it as $f(x) = cx^0$ (unless of course $x = 0$, but we said we would not get hung up on that). So the leading term (highest power of x) in f is x^0. Therefore the degree of f is 0. There is one exception to that last observation: if $c = 0$, then we have the *zero polynomial* $p(x) = 0$. The zero polynomial has no leading term (remember the leading term must have a nonzero coefficient). Therefore its degree is usually not defined, or sometimes defined to be $-\infty$ for reasons that we do not need to delve into now.

A polynomial p with real coefficients can always be regarded as a function $\mathbb{R} \to \mathbb{R}$. This is because for any real number x, $p(x)$ can be evaluted and gives a real number as a result. So p is a rule that assigns to a real number input a real number output.

A number c such that $p(c) = 0$ is called a *root* or *zero* of the polynomial p. The roots of p are exactly those places where the graph of p crosses the x-axis. A useful (and nonobvious) fact about polynomials is that if $p(c) = 0$ then p can expressed as

$$p(x) = (x - c)q(x)$$

for some other polynomial q. In other words, $x - c$ can be factored out. This has to do with the fact that polynomials can be divided (and multiplied) much like integer numbers, using basically the same long-hand division procedure. Chances are you have seen this in one of your algebra courses. If you are drawing a blank at this point, you should really Google polynomial division. For example, Khan Academy has video lectures of how this is done. We are not going to do polynomial division much or at all in this cours, but we will use the consequence about factoring out that I mentioned above. Vice versa, it is also true that if $x - c$ is a factor of the polynomial p, that is $p(x) = (x - c)q(x)$ for some polynomial q, then c is a root of p. This is easy to see by substituting c into p:

$$p(c) = (c - c)q(c) = 0q(c) = 0.$$

Suppose $c_1, c_2, \ldots, c_n$ are the roots of the polynomial p. Then

$$x - c_1, x - c_2, \ldots, x - c_n$$

can all be factored out as

$$p(x) = (x - c_1)(x - c_2) \cdots (x - c_n)q(x)$$

where q is also some polynomial $q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$. If you were to multiply all of the above out using the distributive law, you could choose x from all of the factors of the form $x - c_i$ and the leading term $b_m x^m$ from q and you would get a term $b_m x^{m+n}$. This would be the highest degree term you could get, so this would have to be the leading term of p. So

$$\deg(p) = m + n \geq n.$$

This leads us to the following observation: the number of roots of a polynomial cannot exceed its degree. So a polynomial of degree n has at most n roots, and therefore its graph crosses the x-axis at most n times.

Polynomials are frequently used in mathematical models. We will see them often in this course.

Rational functions are quotients of polynomials. So f is a rational function if

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials and q is not the zero polynomial $q(x) = 0$. The name comes from the analogy with rational numbers. A rational number is a quotient of two integers, while a rational function is a quotient of two polynomials. Note that the denominator $q(x)$ can be 0 for some particular values of x, just not for all of them. In fact, it is quite important where q has a value of 0 because those numbers are not in the domain of f. We will see rational functions in this course from time to time. It is difficult to say anything general about the shapes of their graphs, other than they can have holes and discontinuities.

Trigonometric functions should also be familiar to you. Make sure you understand their definitions in terms of right triangles and the unit circle. You should also be familiar with their graphs. One important note here that whenever we deal with trigonometric functions in calculus, we will always assume that their inputs are angles measured in radians. This is because radians are a more natural way to measure numbers than degrees, despite the fact that the numbers appear to be messier. Formulas in calculus turn out to be nice if the angles are measured in radians, and messy if the angles are in degrees. You have probably seen a number of trigonometric identities in your previous math courses. Let me mention a few of the most important ones to us in this course:

$$\begin{aligned}
\sin(-x) &= -\sin(x) \\
\cos(-x) &= \cos(x) \\
\tan(x) &= \frac{\sin(x)}{\cos(x)} \\
\cos(x) &= \sin\left(x + \frac{\pi}{2}\right)
\end{aligned}$$

for any real number x. All of these can be made sense of using the unit circle interpretation of trigonometric functions. Note that the first two say that sine is an odd function and cosine is an even function.

Our textbook mentions *exponential* and *logarithmic* functions in passing. Exponential and logarithmic functions are also frequent ingredients of models, especially in population dynamics, ecology, epidemiology, economics, and in particular, finance, and in many other sciences and social sciences. Let me remind you of a few things you have likely seen about these functions. The exponential function $f(x) = a^x$, where a is a constant called the *base*, is defined for positive values of a. If $a > 1$, we have a function whose values are increasing as x increases. This should make sense if you think about integer values of x at least. If $0 < a < 1$, then the function is decreasing. If $a = 1$, we juts get the constant function $f(x) = 1$. Negative values of a are problematic because of noninteger values of x. E.g. $(-5)^{3/2}$ does not have a real value as we noted above. Technically, we could set $a = 0$, but we would just get the constant function $f(x) = 0$ except with a whole in the graph at $x = 0$.

We will play with the graphs in class a bit and we will see that they all look similar. The value of a determines the steepness of the graph. You have likely come across the exponential function $f(x) = e^x$ where e is an irrational number $e = 2.7182\ldots$. This is often treated as the mother of all exponential functions. In fact, it is possible to express any exponential function in the form $f(x) = e^{kx}$ for some appropriate number k. The reason for the popularity of the number e despite the fact that it is irrational and hence does not have a particularly nice form as a decimal fraction has to do with calculus. It turns out that the exponential function with this particular base results in simpler formulas in calculus than any of the other choices for the base.

The most important thing to remember about logarithms is that $\log_a(x)$ is the number y such that $a^y = x$. So

$$a^{\log_a(x)} = x \qquad \text{and} \qquad \log_a(a^x) = x.$$

This close relationship between exponential and logarithmic functions is similar to the relationship between power functions and roots. They are inverse functions. That is one undoes what the other does. We will see in class that their graphs are also closely related. The most popular logarithmic function in calculus is $\log_e$, which everybody writes as ln for natural log. We have reviewed some of the properties of exponents. This would be a good time to remind yourself of the most important properties of logarithms, such as

$$\begin{aligned}
\log_a(1) &= 0 \\
\log_a(xy) &= \log_a(x) + \log_a(y) \\
\log_a\left(\frac{x}{y}\right) &= \log_a(x) - \log_a(y) \\
\log_a\left(\frac{1}{x}\right) &= \log_a(x) \\
\log_a(x^y) &= y\log_a(x)
\end{aligned}$$

If these have faded into the fog in your memory by now, try to spend a little quality time looking them up. Here are two non-properties of log function, very popular with students with the $\neq$ sign replaced (erroneously) by $=$.

$$\begin{aligned}
\log_a(x+y) &\neq \log_a(x) + \log_a(y) \\
\log_a(x-y) &\neq \log_a(x) - \log_a(y)
\end{aligned}$$

This section also talks about the following kinds of transformations of graphs of functions

- Vertical shift: $f(x) + c$
- Vertical stretch/compression/reflection: $cf(x)$
- Horizontal shift: $f(x + c)$
- Horizontal stretch/compression/reflection: $f(cx)$

You will want to focus on understanding what these do to the graph and how the geometric effect is related to the value of c. In particular, the horizontal transformations work in ways that are likely the opposite of what you would expect. We will look at examples in class. We will also want to understand what happens if we apply several of these transformations at the same time, such as $af(x) + b$, or $f(ax) + b$, or $f(ax + b)$. Combining vertical transformations is not difficult, but horizontal ones are again a little counterintuitive. We will play with them in class. Find a way to understand them in a way that makes sense to you. These give us ways to construct new functions from the ones we already have by modifying them slightly.

We can also construct new functions by adding, subtracting, multiplying, and dividing functions. We do this by doing the corresponding algebraic operations to their output values. For example, the sum of two functions f and g of real numbers is the function $f+g$ given by the rule $(f+g)(x) = f(x)+g(x)$. When given the input value x, the function $f+g$ assigns to x the output value $f(x)+g(x)$. When you do this, you need to be somewhat careful with the domains of f and g. The function $f+g$ can only have a number x in its domain if x is both in the domain of f and of g. In other words, the domain of $f+g$ is at most the intersection of the domains of f and g. When you divide f/g you also need to be careful to exclude any number x where $g(x)=0$ from the domain of f/g.

Composition of functions is another way we can build new functions from old.

Definition 1. Let $f : A \to B$ and $g : B \to C$ be functions. The *composite* function $g \circ f : A \to C$ is given by

$$g \circ f(x) = g(f(x)).$$

Note that the notation is a little reversed: the way $g \circ f$ calculates its value at x is by applying the function f first to x and then g to the output of f. This is a result of the convention that we write the function in front of the input. In practice, this is not a problem if you write out $g \circ f(x)$ as $g(f(x))$ because the parentheses tell you the correct order of applying the two functions.

You need to be careful with domains and codomains when you compose functions. The definition above assumes that the domain of g is the same as the codomain of f. Of course, there is good reason for this: we want to be sure that any potential output of f can be a meaningful input to g. In practical problems (that is computational problems), it is sometimes advantageous to be a little more flexible: nothing will go wrong as long as the range of f is a subset of the domain of g. The book has a number of examples to familiarize yourself with compositions.

Note that composition of functions is not commutative: $f \circ g \neq g \circ f$ in general. You can see this in Example 5 in our textbook. In fact, depending on the domains and codomains, it could easily happen that one of $f \circ g$ and $g \circ f$ is not even defined, while the other is.

On the other hand, composition of functions is associative: $h \circ (g \circ f) = (h \circ g) \circ f$ for any functions $f : A \to B$, $g : B \to C$, and $h : C \to D$. This is easy to see. First, note that $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are both functions from A to D. Second, if x is any element of A, then

$$h \circ (g \circ f)(x) = h(g \circ f(x)) = h(g(f(x)))$$

and

$$(h \circ g) \circ f(x) = h \circ g(f(x)) = h(g(f(x))),$$

which are the same.