MCS 118 Exam 1 Solutions

1. (10 pts) Determine whether the function

$$f(x) = x|x|$$

is even, odd, or neither. Make sure you justify your answer. (Hint: Since we saw in class that graphs can cheat by appearing to have symmetry they do not actually have, guessing symmetry from the graph is not really convincing justification. Use the definitions of even/odd functions for a convincing argument.)

First, notice that $|-x| = |x|$ for any $x \in \mathbb{R}$ since $-x$ and x are the same distance from 0 on the number line. Now, for any $x \in \mathbb{R}$,

$$f(-x) = -x|-x| = -x|x| = -f(x).$$

Hence f is an odd function.

2. The monthly cost of driving a car depends on the number of miles driven. Lynn (who looks a little dudish in the illustration on the left, but don't get thrown off by this detail) found that in May, it cost her \$380 to drive 480 miles and in June it cost her \$460 to drive 800 miles.

   (a) (4 pts) Express the monthly cost C as a function of the distance driven d assuming that a linear relationship gives a suitable model.

   So $C(d) = md + b$, where m is the slope and b is the vertical intercept. As usual,

$$m = \frac{\Delta C}{\Delta d} = \frac{460 - 380}{800 - 480} = \frac{80}{320} = \frac{1}{4}\,\frac{\$}{\text{mile}}.$$

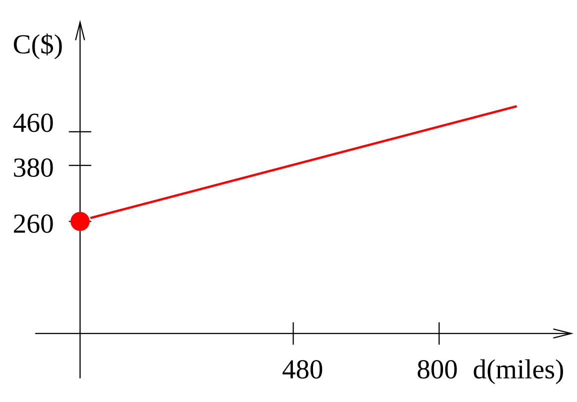
   Now that we know $C = d/4 + b$, we can find the value of b by substituting either of the given points:

$$460 = \frac{800}{4} + b \implies b = 460 - 200 = \$260.$$

   So

$$C(d) = \frac{d}{4} + 260.$$

   (b) (3 pts) Draw the graph of the linear function.



(c) (3 pts) Why does a linear function give a suitable model in this situation? (Hint: a linear function has a constant slope.)

Because it is reasonable enough to think that each additional mile costs the same, i.e. fuel and the average wear and tear on the car is the same for each additional mile driven.

3. (a) (3 pts) State the definition of a function f from a set S to a set T.

A *function* from a set S to a set T is a rule which assigns to every element x in S one and only one element $f(x)$ in T.

(b) (3 pts) What does the Vertical Line Test say?

It says that a curve (graph) in the xy-plane is the graph of a function if and only if no vertical line intersects it more than once.

(c) (4 pts) Explain why a graph that fails the Vertical Line Test cannot be the graph of a function.

A graph that fails the Vertical Line Test is a graph that is intersected more than once by some vertical line. This means there are at least two points on the graph with the same x-coordinate and different y-coordinates. Those points would correspond to the same input value assigned to multiple output values, which a function cannot do.

4. As a young but experienced smuggler working for Jabba the Hutt, Han Solo makes 10000 Imperial credits for successfully smuggling a load of spice (which is an illicit substance) on his Millenium Falcon. If Han smuggles more than five loads of spice in a month, Jabba pays him 15000 credits for each of the next five loads (that is for the 6th through the 10th loads). Things get really lucrative for Han if he manages to smuggle more than ten loads of spice in one month: he gets paid 20000 credits for each load beyond the first the ten.

(a) (4 pts) Construct a piecewise defined function $p(x)$ for Han's smuggling income in a month, where x is the number of loads of spice he smuggled that month. Explain your work.

If Han smuggles between 0 and 5 loads of spice, then he is paid $10000x$ Imperial credits for his work. If he smuggles more than 5 but at most 10 loads of spice, he is paid 50000 credits for the first 5 loads and $15000(x-5)$ credits for the loads beyond 5. So his pay is $50000+15000(x-5)$ in this case. If he smuggles more than 10 loads of spice, then he is paid $50000+75000=125000$ credits for the first 10 loads and $20000(x-10)$ credits for the loads beyond 10. So his pay is $125000+20000(x-10)$ in this case. Hence

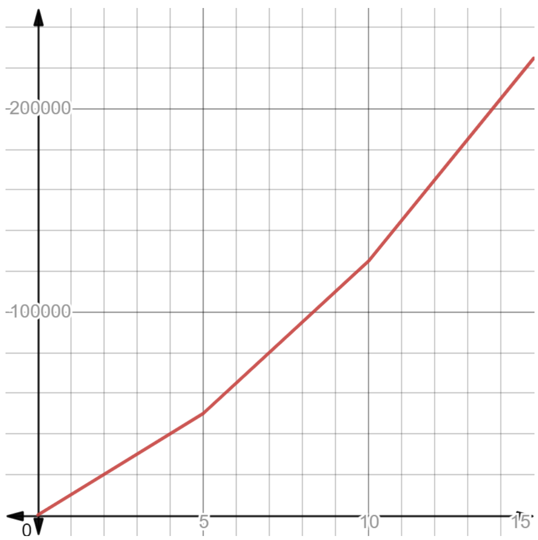
$$p(x)=\begin{cases}10000x & \text{if } 0\le x\le 5\\ 50000+15000(x-5) & \text{if } 5<x\le 10\\ 125000+20000(x-10) & \text{if } 10<x\end{cases}$$

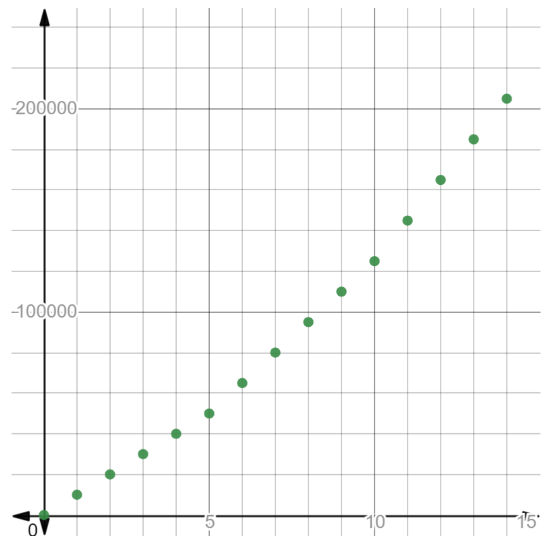
(b) (2 pts) What is the domain of p? Think carefully about this and justify your answer.

For sure, Han cannot smuggle a negative number of loads. He can smuggle no loads at all in a month, e.g. he is on vacation or the Millenium Falcon is down for repairs. When it comes to positive loads, the question is whether he can only smuggle an integer number of loads or he can also smuggle partial loads. This is not really clear from the problem statement and we would have to know more about spice and the practices of its illegal trade to know if this is realistic. The problem does give a slight hint when it says "for the 6th through the 10th loads," which suggests that once Han exceeds 5 loads in a month, the next realistic number is 6 loads, that is the number of loads is integer. So the domain of p could be either the set of nonnegative integers, $\mathbb{Z}^{\geq 0}$, or the set of nonnegative real numbers, $\mathbb{R}^{\geq 0}$. Either is a reasonable answer.

(c) (4 pts) Graph the function p.

The graph depends on whether the domain is $\mathbb{Z}^{\geq 0}$ or $\mathbb{R}^{\geq 0}$. If it is the latter, the graph (on the left) consists of three linear pieces. If the domain is $\mathbb{Z}^{\geq 0}$, the graph (on the right) includes only those points on these lines whose x-coordinates are integer.





5. (5 pts each) **Extra credit problem.** Let $f : \mathbb{R} \to \mathbb{R}$ be an even function and let $g(x) = f(x) + 1$.

(a) Does g have to be an even function? If you think it does, give a proof; if you do not think it does, find a counterexample.

Since f is even, we know $f(-x) = f(x)$ for all possible values of x, that is for all $x \in \mathbb{R}$. Hence

$$g(-x) = f(-x) + 1 = f(x) + 1 = g(x)$$

is also true for all $x \in \mathbb{R}$. It follows that g does have to be to an even function.

Here is another way to think of this in terms of reflections. We know the graph of f must be symmetric with respect to reflection across the y-axis. The graph of g is the same as the graph of f except it is shifted up by 1 unit. Since the shift is parallel to the axis of symmetry, the graph of g must still be symmetric about the y-axis.

(b) Is it possible for g to be an odd function? If you think it is, find an example; if you do not think it is, find an argument to show it is not possible.

For g to be odd, it would have to be true that $g(-x) = -g(x)$ for all values of x. We already know $g(-x) = g(x)$. So

$$g(x) = -g(x) \implies 2g(x) = 0 \implies g(x) = 0$$

for all $x \in \mathbb{R}$. This is possible, if $f(x) = -1$ for all $x \in \mathbb{R}$. Note that this function f is still even because $f(-x) = -1 = f(x)$ for all $x \in \mathbb{R}$. So it is in fact possible for f to be an even function and $g(x) = f(x) + 1$ to be an odd function, although only in this one case.