MCS 118 Exam 2 Solutions

1. (10 pts) Let $f(x) = \frac{x}{1+x}$. Find the correct expression for the function $f \circ f(x)$ and the largest possible subset of the real numbers that could be the domain of $f \circ f$. Do not forget to justify your answers.

$$f \circ f(x) = f(f(x)) = \frac{f(x)}{1+f(x)} = \frac{\frac{x}{1+x}}{1+\frac{x}{1+x}}$$

Note that if $1 + x = 0$, that is when $x = -1$, then we have 0 in the denominator of $\frac{x}{1+x}$. So $x = -1$ must be excluded from the domain. In fact, whenever $x \neq -1$, we can simplify $f(x)$ as

$$f \circ f(x) = \frac{\frac{x}{1+x}}{1+\frac{x}{1+x}} = \frac{\frac{x}{1+x}(1+x)}{\left(1+\frac{x}{1+x}\right)(1+x)} = \frac{x}{1+x+x} = \frac{x}{1+2x}.$$

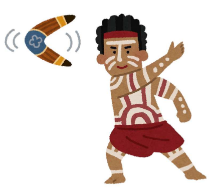
Since the denominator $1 + 2x$ would be 0 if $x = -1/2$, we must also exclude $-1/2$ from the domain of $f \circ f$. For any other value of x, the fractions $\frac{x}{1+x}$ and $\frac{x}{1+2x}$ have legitimate values. So the largest possible subset of the real numbers that can be the domain of $f \circ f$ is

$$\mathbb{R} \setminus \left\{-1, -\frac{1}{2}\right\} = (-\infty, -1) \cup \left(-1, -\frac{1}{2}\right) \cup (-1/2, \infty).$$

2. (10 pts) Let f and g be linear functions with equations $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$. Is $f \circ g$ also a linear function? If so, what is the slope of its graph? Do not forget to justify your answers.

$$f \circ g(x) = m_1 g(x) + b_1 = m_1(m_2x + b_2) + b_1 = m_1m_2x + m_1b_2 + b_1$$

This is again of the form $mx + b$ for $m = m_1m_2$ and $b = m_1b_2 + b_1$. Hence it is a linear function. Its slope is $m = m_1m_2$.



3. An Australian aboriginee throws a boomerang parallel to the ground. The distance of the boomerang from its owner at time t is given by the function $d(t) = -4t^2 + 30t$, where the distance is measured in meters and time is seconds. The boomerang barely misses a peacefully grazing kangaroo at $t = 2$.

(a) (6 pts) Find the average velocity of the boomerang over the following time intervals:

(i) $[2, 2.5]$

$$v_{\text{ave}} = \frac{\Delta d}{\Delta t} = \frac{d(2.5) - d(2)}{2.5 - 2} = \frac{-4(2.5)^2 + 30(2.5) - (-4(2)^2 + 30(2))}{0.5}$$
$$= \frac{50 - 44}{0.5} = \frac{6}{0.5} = 12 \text{ m/s}$$

(ii) $[1.9, 2]$

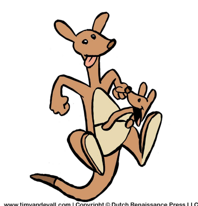
$$v_{\text{ave}} = \frac{\Delta d}{\Delta t} = \frac{d(2) - d(1.9)}{2 - 1.9} = \frac{-4(2)^2 + 30(2) - (-4(1.9)^2 + 30(1.9))}{0.1}$$
$$= \frac{44 - 42.56}{0.1} = \frac{1.44}{0.1} = 14.4 \text{ m/s}$$

(iii) $[2, 2.05]$

$$v_{\text{ave}} = \frac{\Delta d}{\Delta t} = \frac{d(2.05) - d(2)}{2.05 - 2} = \frac{-4(2.05)^2 + 30(2.05) - (-4(2)^2 + 30(2))}{0.05}$$
$$= \frac{44.69 - 44}{0.05} = \frac{0.69}{0.05} = 13.8 \text{ m/s}$$

(iv) $[1.99, 2]$

$$v_{\text{ave}} = \frac{\Delta d}{\Delta t} = \frac{d(2) - d(1.99)}{2 - 1.99} = \frac{-4(2)^2 + 30(2) - (-4(1.99)^2 + 30(1.99))}{0.01}$$
$$= \frac{44 - 43.8596}{0.01} = \frac{0.1404}{0.01} = 14.04 \text{ m/s}$$



(b) (4 pts) Use your results in part (a) to estimate the instantaneous velocity of the boomerang as it passes by the startled kangaroo at $t = 2$. Explain how you came up with your estimate.

The numbers in part (a) seem to be approaching 14 m/s. Of course, we cannot really tell if they are getting close to 14, or some other number that is close to 14. We could calculate more values over ever shorter intervals to be able to tell how close they are getting to 14. Based on the four values in part (a), I would place my bet on 14 m/s for the instantaneous velocity.
By the way, as a bit of reality check, 14 m/s = 50.4 km/s or about 32 miles/h, which seems reasonable as the speed of a flying boomerang.

4. (a) (4 pts) Let f be a function of real numbers and $a \in \mathbb{R}$ such that in some neighborhood of a, $f(x)$ has a value for x except possibly at $x = a$. State the informal definition of

$$\lim_{x \to a} f(x) = L$$

where L is a real number.

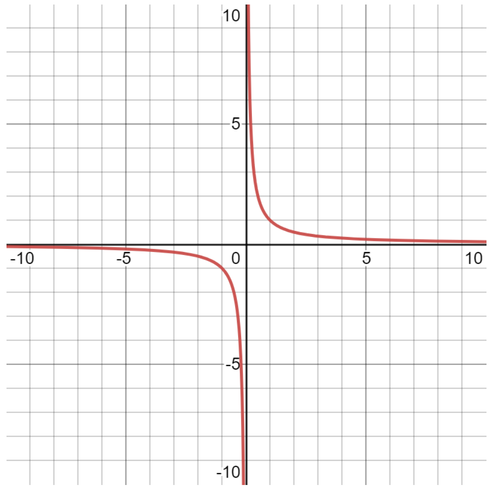
This is Definition 1 in Section 1.3. You can see the textbook's version there. This is how we said it in class:

$$\lim_{x \to a} f(x) = L$$

means that the value of $f(x)$ can be made arbitrarily close (as close as you want) to L if x is made sufficiently close to a but not equal to a.

(b) (6 pts) Explain why the function $f(x) = \frac{1}{x}$ does not have a limit as $x \to 0$.

When x is close to 0, the value of $1/x$ is either a large positive number (if x is positive) or a negative number of large magnitude (if x is negative). The closer x gets to 0, the larger the magnitude of $1/x$ is, while the sign of $1/x$ depends on the sign of x. In any case, $1/x$ does not get close to any particular number as x gets closer and closer to 0. This is quite clear from the graph of $1/x$:



5. (10 pts) **Extra credit problem.** Remember that a rational number is a number of the form m/n where m and n are integers and $n \neq 0$, and an irrational number is a real number that cannot be expressed as a ratio of two integers. For example, $2/3$ and $-13/5$ are rational numbers, but π and $\sqrt{2}$ are known to be irrational. One property (not essential for answering the question below) of rational/irrational numbers you may have encountered is that rational numbers have finite or infinite but repeating decimal expansions, whereas irrational numbers have infinite decimal expansions without a repeating pattern.

   Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number} \\ -1 & \text{if } x \text{ is an irrational number} \end{cases}.$$

   Does either

$$\lim_{x \to 0^+} f(x)$$

   or

$$\lim_{x \to 0^-} f(x)$$

   exist? If so, find its value; if it does not, explain why it does not exist.

   Neither one-sided limit exists. This is because no matter how close we look to 0, there are boht rational and irrational numbers there. In other words, no matter how close x gets to 0, x could be either rational or irrational, and hence the value of $f(x)$ could be 1 or -1. This is true on both sides of 0. For example, $\frac{1}{n}$, where n is an integer, is a rational number which can be made to be as close to 0 (and either positive or negative, depending on the sign of n) as we want by making the magnitude of n large; while $\frac{\sqrt{2}}{n}$, where n is an integer, is an irrational number which can be made to be as close to 0 (and either positive or negative) as we want by making the magnitude of n large.

   By the way, how do we know $\frac{1}{n}$ is a rational number? It is a ratio of integers. And how do we know $\frac{\sqrt{2}}{n}$ is irrational? It cannot be expressed as a ratio of two integers because if it could, then $\frac{\sqrt{2}}{n} = \frac{k}{m}$ for some integers k and m and hence $\sqrt{2} = \frac{nk}{m}$, which would make $\sqrt{2}$ a ratio of the integers nk and m, and hence $\sqrt{2}$ would have to be a rational number. But we know $\sqrt{2}$ is not a rational number.