MCS 118 Final Exam

Dec 15, 2020

All of your answers must be carefully justified. Neat work, clear and to-the-point explanations will receive more credit than messy, chaotic answers. You may refer to any result proved in class unless otherwise specified. You may use results you proved on your homework, except for ones the problem specifically asks you to prove.

You are not allowed to use your textbook or your class notes, but you may use a simple calculator.

1. (5 pts each)
   (a) Determine whether the function $f : \mathbb{R} \to \mathbb{R}$ given by
   $$f(x) = \frac{x^2}{x^4 + 1}$$
   is even, odd, or neither. Why?

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. If f is an even function and g is an odd function, what can you say about the symmetry of fg? Is fg even or odd or both or neither in this case? Do not forget to justify your answer.

2. (10 pts) Let
   $$f(x) = \frac{x}{1+x} \quad \text{and} \quad g(x) = \sin(2x).$$
   Find $f \circ g$ and the largest possible subset of the real numbers that could be the domain of $f \circ g$.

3. (10 pts) Evaluate
   $$\lim_{u \to -2} \sqrt{u^4 + 3u + 6}$$
   and justify each step by indicating the appropriate Limit Law(s) you used.

4. (a) (4 pts) Let f be a function whose inputs and outputs are real numbers. Define what it means for f to be a decreasing function.

   (b) (6 pts) Give an example of a nonlinear function that is decreasing. Justify your example.

5. (10 pts) Use the formal definition of the limit to prove that
   $$\lim_{x \to -3} (4x + 10) = -2.$$

6. Let c be a real number and define the function $f : \mathbb{R} \setminus \{-1\} \to \mathbb{R}$ by
   $$f(x) = \begin{cases} x^2 - cx + 5 & \text{if } x < -2 \\ \frac{c}{x+1} & \text{if } x \geq -2. \end{cases}$$

   (a) (4 pts) Find
   $$\lim_{x \to -2^+} f(x)$$
   and fully justify your answer.

   (b) (3 pts) Find
   $$\lim_{x \to -2^-} f(x)$$
   and fully justify your answer.

(c) (3 pts) For what value of c does

$$\lim_{x \to -2} f(x)$$

exist? Why?

7. (10 pts) **Extra credit problem.** We have learned that Limit Law 4 says if

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

both exist then

$$\lim_{x \to a} [f(x)g(x)]$$

also exists, and in fact,

$$\lim_{x \to a} [f(x)g(x)] = \left[\lim_{x \to a} f(x)\right] \left[\lim_{x \to a} g(x)\right].$$

Find an example of two functions f and g such that neither

$$\lim_{x \to a} f(x) \quad \text{nor} \quad \lim_{x \to a} g(x)$$

exists, yet

$$\lim_{x \to a} [f(x)g(x)]$$

does exist. Justify your example.