MCS 118 Final Exam Solutions

1. (5 pts each) Is f a function from S to T? Why or why not? Do not forget to fully justify your answer.

   (a)

$$\begin{aligned} S &= \text{set of current Gustavus students} \\ T &= \text{set of dorms at GAC} \\ f(x) &= \text{the dorm } x \text{ lives in} \end{aligned}$$

   This is not a function because there are some Gusties who live off campus, and if x is such a student, then f cannot assign a value in T to x.

   (b)

$$\begin{aligned} S &= \text{set of people who have siblings} \\ T &= \text{set of people who have ever lived} \\ f(x) &= x\text{'s oldest sibling} \end{aligned}$$

   This is a function because anyone who has at least one sibling also has an oldest sibling. Just sort x's siblings by age and the oldest one is the oldest sibling. It does not make sense to argue that if x is the oldest among the children in x's generation, then x does not have an oldest sibling. Remember that x is not x's sibling.

   Alternately, a reasonable argument can be made that f may not be a function because it is possible that someone could have two or more oldest siblings. For example, they could be twins (or multiplets) that were really born at the same time. Or siblings could include half-siblings or step siblings, and it is possible (although not very likely) that someone could have two oldest half-siblings or step siblings that were born at the exactly the same time.

2. (5 pts each) Let f and g be functions of real numbers such that f can be composed with g. Suppose g is an odd function and let $h = f \circ g$.

   (a) Is h always an odd function? If you think it is, prove that it is; if you do not think it is, find a counterexample.

   No, h is not always odd. Here is a counterexample. Let $f, g : \mathbb{R} \to \mathbb{R}$ be the following functions

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= x \\ h(x) &= f \circ g(x) = f(g(x)) = x^2 \end{aligned}$$

   Note that $g(-x) = -x = -g(x)$ for every real number x, so g is indeed an odd function. But h is not odd because

$$h(-x) = (-x)^2 = x^2 \neq -x^2 = -h(x)$$

   in general. For example, $h(-1) = 1$ and $-h(1) = -1$ are not equal.

   (b) What if f is even? Is h always an odd or an even function then? Or can it be neither? If you think it is always odd or always even, prove that it is; if you do not think it is, find a counterexample.

Let x be any real number such that $h(-x)$ has a value. Then

$$\begin{aligned} h(-x) &= f(g(-x)) \\ &= f(-g(x)) && \text{since } g(-x) = -g(x) \\ &= f(g(x)) && \text{since } f(-y) = f(y) \text{ for } y = g(x) \\ &= h(x) \end{aligned}$$

We have shown that $h(-x) = h(x)$ for every x in the domain of h.

3. (10 pts) Evaluate the limit if it exists:

$$\lim_{x \to -1} \frac{x^2 + 2x + 1}{x^4 - 1}$$

Do not forget to carefully justify each step in your calculation.

First, notice that LL5 will not work because $\lim_{x \to -1}(x^4 - 1) = 0$ by direct subsitution into the polynomial. But we can factor the numerator and the denominator:

$$\frac{x^2 + 2x + 1}{x^4 - 1} = \frac{(x+1)^2}{(x+1)(x^3 - x^2 + x - 1)}.$$

We can now cancel $x + 1$ as long as $x + 1 \neq 0$. Actually, we know $x + 1 \neq 0$ because $x + 1$ would be 0 if $x = -1$, but in the limit, as $x \to -1$, x is close to -1, but $x \neq -1$. So

$$\lim_{x \to -1} \frac{x^2 + 2x + 1}{x^4 - 1} = \lim_{x \to -1} \frac{(x+1)^2}{(x+1)(x^3 - x^2 + x - 1)} = \lim_{x \to -1} \frac{x+1}{x^3 - x^2 + x - 1}.$$

Since the function in the limit is a rational function, we can evaluate the limit by direct substitution:

$$\lim_{x \to -1} \frac{x^2 + 2x + 1}{x^4 - 1} = \lim_{x \to -1} \frac{x+1}{x^3 - x^2 + x - 1} = \lim_{x \to -1} (\frac{-1+1}{(-1)^3 - (-1)^2 + (-1) - 1} = \frac{0}{-4} = 0.$$

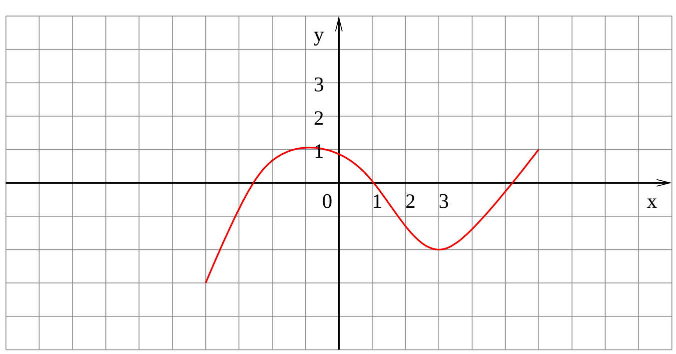
We also could have used Theorem 2 from Section 1.4 to say that because

$$\frac{x^2 + 2x + 1}{x^4 - 1} = \frac{(x+1)^2}{(x+1)(x^3 - x^2 + x - 1)}$$

for every $x \neq -1$,

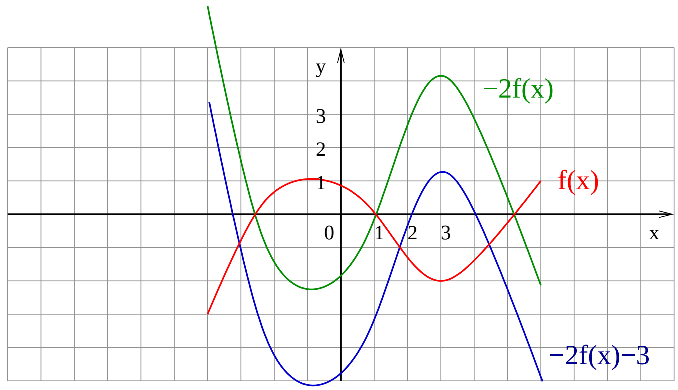
$$\lim_{x \to -1} \frac{x^2 + 2x + 1}{x^4 - 1} = \lim_{x \to -1} (\frac{x+1}{x^3 - x^2 + x - 1}.$$

4. (5 pts each) The graph of a function $f(x)$ is given below.



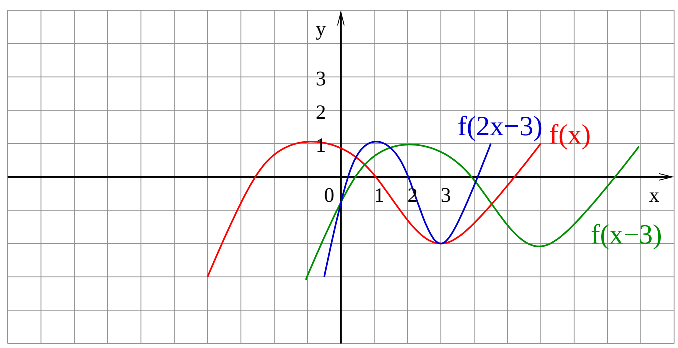
Sketch the graphs of the following functions. Make sure you explain your work.

(a) $g(x) = -2f(x) - 3$



Subtracting 3 from the output variable shifts the graph down by 3 units, multiplying the output by -2 stretches the graph vertically by a factor of 2 and reflects it across the x-axis. The order of these transformations does matter, the stretch/reflection must be done before the shift.

(b) $h(x) = f(2x - 3)$



Subtracting 3 from the input variable shifts the graph right by 3 units, multiplying the input by 2 compresses the graph horizontally by a factor of 2. In this case, the shift comes before the compression.

5. (a) (4 pts) Let f be a function of real numbers and $a \in \mathbb{R}$ such that in some neighborhood of a, $f(x)$ has a value for x except possibly at $x = a$. State the formal (precise) definition of

$$\lim_{x \to a} f(x) = L$$

where L is a real number.

That

$$\lim_{x \to a} f(x) = L$$

means that for every $\epsilon > 0$ there exists a corresponding $\delta > 0$ such that if

$$0 < |x - a| < \delta$$

then

$$|f(x) - L| < \epsilon.$$

(b) (6 pts) Find
$$\lim_{x\to 5} |x-5|^3.$$
You do not need to use the formal definition of the limit. Use whatever tools you need to find the limit, but be sure to give a rigorous argument to justify your work. (Hint: look at one-sided limits.)

We will look at the one-sided limits.
$$\lim_{x\to 5^+} |x-5|^3 = \lim_{x\to 5^+} (x-5)^3$$
because when $x > 5$ then $x-5>0$ and hence $|x-5| = x-5$. Now, $(x-5)^3$ is a polynomial, so its limit can be found by direct substitution:
$$\lim_{x\to 5^+} (x-5)^3 = (5-5)^3 = 0^3 = 0.$$
As for the left limit,
$$\lim_{x\to 5^-} |x-5|^3 = \lim_{x\to 5^-} (5-x)^3$$
because when $x<5$ then $x-5<0$ and hence $|x-5| = -(x-5) = 5-x$. Since $(5-x)^3$ is a polynomial, we can find its limit by direct substitution:
$$\lim_{x\to 5^-} (5-x)^3 = (5-5)^3 = 0^3 = 0.$$
Since the left and right limits are both 0, the two-sided limit is also 0:
$$\lim_{x\to 5} |x-5|^3 = 0$$

6. (10 pts) Let $p(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$ be a polynomial function with real coefficients, that is $b_i \in \mathbb{R}$ for every $i = 0, 1 \ldots, n$. Prove that p has the direct substitution property
$$\lim_{x\to a} p(x) = p(a)$$
where a is any real number.

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a}(b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0) \\
&= \lim_{x\to a}(b_n x^n) + \lim_{x\to a}(b_{n-1}x^{n-1}) + \cdots + \lim_{x\to a}(b_1 x) + \lim_{x\to a} b_0 && \text{by LL1} \\
&= b_n \lim_{x\to a} x^n + b_{n-1} \lim_{x\to a} x^{n-1} + \cdots + b_1 \lim_{x\to a} x + \lim_{x\to a} b_0 && \text{by LL3} \\
&= b_n a^n + b_{n-1}a^{n-1} + \cdots + b_1 a + b_0 && \text{by LL9 and LL7} \\
&= f(a).
\end{aligned}$$

7. **Extra credit problem.** The purpose of this problem is to prove Limit Law 4: if f and g are functions of real numbers and $a \in \mathbb{R}$ such that
$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$
both exist, then
$$\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x).$$
I will break this down into steps.

(a) (10 pts) First, we will prove that the limit law holds in the special case when

$$\lim_{x\to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a} g(x) = 0.$$

So suppose

$$\lim_{x\to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a} g(x) = 0,$$

and use the formal definition of the limit (that is δ's and ϵ's) to prove that

$$\lim_{x\to a} [f(x)g(x)] = 0.$$

Hint: one way to do this is to find a good value of $\delta > 0$ such that if $0 < |x - a| < \delta$ then both $f(x)$ and $g(x)$ are closer to 0 than a distance of $\sqrt{\epsilon}$.

We need to prove that for every $\epsilon > 0$ there is a corresponding $\delta > 0$ such that if

$$0 < |x - a| < \delta$$

then

$$|f(x)g(x) - 0| < \epsilon.$$

Note that $|f(x)g(x) - 0| = |f(x)g(x)| = |f(x)||g(x)|$. So to make

$$|f(x)g(x) - 0| < \epsilon,$$

we can make $|f(x)| < \sqrt{\epsilon}$ and $|g(x)| < \sqrt{\epsilon}$:

$$|f(x)g(x) - 0| = |f(x)g(x)| = |f(x)||g(x)| < \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon.$$

Since

$$\lim_{x\to a} f(x) = 0$$

we know there exists some $\delta_1 > 0$ such that if

$$0 < |x - a| < \delta_1$$

then

$$|f(x) - 0| < \sqrt{\epsilon}.$$

Similarly,

$$\lim_{x\to a} g(x) = 0$$

tells us there exists some $\delta_2 > 0$ such that if

$$0 < |x - a| < \delta_2$$

then

$$|g(x) - 0| < \sqrt{\epsilon}.$$

This suggests that we should let $\delta = \min(\delta_1, \delta_2)$.
Suppose

$$0 < |x - a| < \delta.$$

Then

$$0 < |x - a| < \delta_1 \qquad \text{and} \qquad 0 < |x - a| < \delta_2$$

are both true. Hence

$$|f(x)g(x) - 0| = |f(x)g(x)| = |f(x)||g(x)| = |f(x) - 0||g(x) - 0| < \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon.$$

(b) (5 pts) Now, if

$$\lim_{x \to a} f(x) = L_1 \qquad \text{and} \qquad \lim_{x \to a} g(x) = L_2$$

then

$$\lim_{x \to a}(f(x) - L_1) = \lim_{x \to a} f(x) - \lim_{x \to a} L_1 = L_1 - L_1 = 0$$

and

$$\lim_{x \to a}(g(x) - L_2) = \lim_{x \to a} g(x) - \lim_{x \to a} L_2 = L_2 - L_2 = 0$$

by Limit Laws 2 and 7. So the functions $F(x) = f(x) - L_1$ and $G(x) = g(x) - L_2$ satisfy

$$\lim_{x \to a} F(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} G(x) = 0.$$

Hence by part (a),

$$\lim_{x \to a}[F(x)G(x)] = 0.$$

Use that $f(x) = F(x) + L_1$ and $g(x) = G(x) + L_2$ and the limit laws (other than LL4) to show that

$$\lim_{x \to a}[f(x)g(x)] = L_1 L_2.$$

$$\begin{aligned}
\lim_{x \to a}[f(x)g(x)] &= \lim_{x \to a}[(F(x) + L_1)(G(x) + L_2)] \\
&= \lim_{x \to a}[F(x)G(x) + F(x)L_2 + L_1 G(x) + L_1 L_2] \\
&= \lim_{x \to a}[F(x)G(x)] + \lim_{x \to a}[F(x)L_2] + \lim_{x \to a}[L_1 G(x)] + \lim_{x \to a}[L_1 L_2] && \text{by LL1} \\
&= \lim_{x \to a}[F(x)G(x)] + L_2 \lim_{x \to a} F(x) + L_1 \lim_{x \to a} G(x) + L_1 L_2 && \text{by LL3 and LL7}
\end{aligned}$$

We can now use that

$$\begin{aligned}
\lim_{x \to a} F(x) &= 0 \\
\lim_{x \to a} G(x) &= 0 \\
\lim_{x \to a}[F(x)G(x)] &= 0
\end{aligned}$$

to conclude

$$\lim_{x \to a}[f(x)g(x)] = 0 + L_2 \cdot 0 + L_1 \cdot 0 + L_1 L_2 = L_1 L_2.$$