Notes for Section 1.3

Limits are probably the most conceptually difficult part of a Calculus 1 course. The motivating example (Example 1) is well worth paying attention to because it foreshadows why limits are important to us. One of the main themes of calculus is the instantaneous rate of change of functions. A particular example of that concept is the instantaneous rate of change of position with respect to time, that is *instantaneous velocity* (or speed). If $s(t)$ is the position of an object moving along a straight line, then the important formula here is that

$$\text{average velocity between } t_1 \text{ and } t_2 = \frac{s(t_2) - s(t_1)}{t_2 - t_1},$$

or

$$v_{\text{average}} = \frac{\Delta s}{\Delta t}$$

where $\Delta s = s(t_2) - s(t_1)$ is the change in position (or displacement) and $\Delta t = t_2 - t_1$ is the change in time (or time elapsed). Note that average velocity is always between two points in time.

If we make the amount of time that elapses small, then the velocity of the moving object will not change much between t_1 and t_2, and the average velocity will be close to the instantaneous velocity at t_1 (and also at t_2). So what we want to do is look at what number

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

is getting close to as $t_2 - t_1$ is getting closer and closer to 0. Note that we cannot just let $t_1 = t_2$ because then we would have 0 in the denominator and the value of the fraction would be undefined. So we really need to look at what is happening to the value of

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

as we make t_1 and t_2 get closer and closer but they remain different numbers. As Example 1 illustrates, it is not unreasonable to expect that there could be a specific number that

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

approaches as $t_2 - t_1$ approaches 0.

The ratio

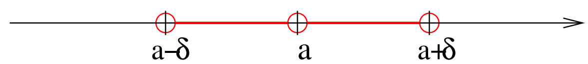
$$\frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

is called the *difference quotient* as it the is the quotient of the difference in position between t_1 and t_2 and the difference in time between t_1 and t_2. Such a number is called the limit of the difference quotient. While the limit of the difference quotient is of great importance in calculus, we will first want to acquaint ourselves with the limit of a simpler function $f(x)$. What we want to study is what happens to the value of $f(x)$ as the input variable x gets closer and closer to a number a. For the next motivating example, think about what happens to the value of $f(x) = x^2$ as x approaches 2. It is reasonable to say that if x is a number close to 2, then x is a number close to 4, and in fact, the closer x gets to 2, the closer x^2 gets to 4. We are relying on intuition here rather than a precise understanding of limits. Let us make this idea a little more exact in the following definition.

Informal definition 1. Let f be a function of real numbers and let $a \in \mathbb{R}$ such that $f(x)$ is defined for every x in some neighborhood of a, except possibly at a. Then we say that the number L is the *limit of $f(x)$ as x approaches a* if the value of $f(x)$ can be made arbitrarily close (as close as you want) to L if x is made sufficiently close to a but not equal to a. If so, we write

$$\lim_{x \to a} f(x) = L.$$

Let us make sense of what this definition says. First, what is meant be a neighborhood of a? What we mean is the numbers that are close to a on either side of a on the number line. How close is close? It does not really matter, but there still has to be some interval around a that contains them. We can say that we mean the numbers between $a-\delta$ and $a+\delta$ where δ is some small positive number. So x in the above informal definition is a number $a-\delta < x < a+\delta$, except $x \neq a$. We could also say this as $x \in (a-\delta, a) \cup (a, a+\delta)$.



The reason we need this is because we want to talk about what happens to the value of $f(x)$ if x is near a on the number line, but then we need to insist that $f(x)$ should have a value in the first place.

The crucial part of this definition is the arbitrarily close/sufficiently close pair. Let me illustrate what these mean via an example. We argued (quite informally) earlier that

$$\lim_{x\to 2} x^2 = 4.$$

Here is what this means. We can ask if we can make x^2 close enough to 4 to be between 3.9 and 4.1, that is x^2 would have be within a distance of 0.1 of 4. In fact, this can be done. If $1.99 < x < 2.01$ then

$$1.99^2 < x^2 < 2.01^2$$

because the function $f(x) = x^2$ is increasing on $x \in \mathbb{R}^{\geq 0}$, so since $1.99 < x < 2.01$, it is also true that $f(1.99) < f(x) < f(2.01)$, which is exactly

$$1.99^2 < x^2 < 2.01^2.$$

Now, $1.99^2 = 3.9601$ and $2.01^2 = 4.0401$. And indeed, as long as $1.99 < x < 2.01$, we have

$$3.9 < 3.9601 < x^2 < 4.0401 < 4.1.$$

So it is not hard to make x^2 fall between 3.9 and 4.1, as long as x is close enough to 2 to be between 1.99 and 2.01. Do not worry too much about how I came up with 1.99 and 2.01 for now. Focus on understanding why they work. Notice that arbitrary and sufficient are not the same thing and they are not interchangeable.

But the definition says we should be able to make x^2 as close as we want to 4. So a distance of 0.1 from 4 may not satisfy us. What if we wanted x^2 to be within 0.01 of 4, that is $3.99 < x^2 < 4.01$? Can we make x get sufficiently close to 2 to guarantee that $3.99 < x^2 < 4.01$? We can do this too. Notice that if

$$1.999 < x < 2.001$$

then

$$1.999^2 < x^2 < 2.001^2.$$

Now, $1.999^2 = 3.996001$ and $2.001^2 = 4.004001$. And indeed, as long as $1.999 < x < 2.001$, we have

$$3.99 < 3.996001 < x^2 < 4.004001 < 4.01.$$

You can guess that we can keep playing this game forever. The tighter we make the bounds around 4 for x^2 the tighter we need to make the bounds around 2 for x, but we can always do this. The important thing for you to understand at this point is not how the game can be played but what the rules of the game are.

You might at this point be thinking why bother doing this when we could just substitute $x = 2$ into $f(x) = x^2$ and note that $f(2) = 4$ and be done with the whole thing. One of the problems with that idea is that depending on what f is, $f(a)$ may not even exist. This is what Example 2 illustrates.

So how can we calculate limits? In Example 1, you saw a table of values you could easily construct with a calculator or a spreadsheet program on a computer. The table of values showed a fairly clear trend and allowed us to guess that the limit should be 49. But notice that this is just a guess. How do we know that it is not 48.999999 or 49.000001? Just because 49 has a nicer decimal expansion than those two numbers does not mean 49 must be the limit. Clearly, calculating values by calculator and guessing the limit based on what those values appear to get close to is not a really reliable way to calculate limits.

Here is another example of how calculating tables of values by using specific numbers can fool you about the value of a limit. Let

$$f(x) = x^2 + \frac{1}{1000000000},$$

and let us try to guess the limit of f as x approaches 0. Here is a table of values:

x	$f(x)$
0.1	0.010 000 001 0
0.01	0.000 100 001 0
0.001	0.000 001 001 0
0.0001	0.000 000 011 0
0.000 05	0.000 000 003 5
−0.1	0.010 000 001 0
−0.01	0.000 100 001 0
−0.001	0.000 001 001 0
−0.0001	0.000 000 011 0
−0.000 05	0.000 000 003 5

Based on this table, it would be easy to conclude that the limit must be 0 since the values of $f(x)$ seem to be getting closer and closer to 0 as x approaches 0. But that would be wrong. In fact, we can tell that x^2 will continue to get closer and closer to 0 as x gets closer to 0, but the $\frac{1}{1000000000} = 0.000000001$ will remain the same. So in fact, the value of this limit is actually 0.000000001.

Example 5 illustrates another pitfall in guessing limits by calculating values at specific numbers. Many people would be content that 0.1, 0.01, 0.001, etc is a good sequence of numbers approaching 0 and the values of f at these numbers should indicate where $f(x)$ is headed as x approaches 0. The trend is clear if $f(x) = \sin(\pi/x)$. But that guess would be wrong. In fact, the example also illustrates that there does not have to be a particular number the values of $f(x)$ get close to as x approaches a. The values of $f(x) = \sin(\pi/x)$ fluctuate between -1 and 1 as x approaches 0. The limit of a function need not exist at a number a. Example 6 shows you another function whose limit does not exist at 0.

Another way you can guess the limit of a function at a point a is to guess where the values of $f(x)$ are going as x approaches a from the graph of f. This is illutrated in Examples 3 and 4. This is not a really reliable and accurate way to find the limit of a function either, as Example 3 explains. Graphs, much like tables of values, are imperfect experimental devices and can easily lead you astray. Nevertheless, both tables of values and graphs are useful tools to gather intuition about the behavior of a function, you just need to be careful to not mistake intuition for fact.

Example 4 also illustrates that "talking your way" through a limit, as we did above with $f(x) = x^2$ may not work. It was easy enough to see that if x was a number close to 2 then x^2 should be a number close to 4 and that the closer x would get to 2, the closer x^2 should get to 4. But it is not at all clear what a function like $f(x) = \frac{\sin(x)}{x}$ should be getting close to as x approaches 0. The graph gives a hint, and the hint is correct in this case.

Example 6 introduces one-sided limits. It is easy to see from the graph why the limit of the Heaviside function $H(t)$ does not exist as $t \to 0$: no matter how close t is to 0, it could be either negative or positive and the value of $H(t)$ could be either 0 or 1 accordingly. Another way to look at this is that if t is moving closer and closer to 0, $H(t)$ could be 0 or 1 or jump back and forth between 0 and 1 as t jumps back and forth, not necessarily in an alternating manner either, between negative and positive values. For example, if t were to approach 0 by taking on the values

$$0.01, 0.005, -0.00022, 0.000037, -0.000019, -0.000008, -0.000001, 0.00000003, \ldots$$

then the corresponding values of $H(t)$ would be

$$1, 1, 0, 1, 0, 0, 0, 1, \ldots,$$

which is not a sequence of numbers that is getting close to any particular number.

On the other hand, if t were to approach 0 while remaining always positive, the corresponding value of $H(t)$ would always be 1, and we could say that the value $H(t)$ approaches 1 as t approaches 0 from the right. Similarly, if t were to approach 0 while remaining always negative, the corresponding value of $H(t)$ would always be 0, and we could say that the value $H(t)$ approaches 0 as t approaches 0 from the left. This kind of behavior motivates the following informal definitions of left and right limits:

Informal definition 2. Let f be a function of real numbers and let $a \in \mathbb{R}$ such that $f(x)$ is defined for every x in some interval $(a, a+\delta)$ where δ is some positive number. Then we say that the number L is the *right limit of $f(x)$ as x approaches a* or the *limit of $f(x)$ as x approaches a from the right* if the value of $f(x)$ can be made arbitrarily close (as close as you want) to L if $x > a$ is made sufficiently close to a. If so, we write

$$\lim_{x \to a^+} f(x) = L.$$

Informal definition 3. Let f be a function of real numbers and let $a \in \mathbb{R}$ such that $f(x)$ is defined for every x in some interval $(a-\delta, a)$ where δ is some positive number. Then we say that the number L is the *left limit of $f(x)$ as x approaches a* or the *limit of $f(x)$ as x approaches a from the left* if the value of $f(x)$ can be made arbitrarily close (as close as you want) to L if $x < a$ is made sufficiently close to a. If so, we write

$$\lim_{x \to a^-} f(x) = L.$$

Note the similarities between these definitions and the informal definition of regular (or two-sided) limits. Compare them and notice what the differences are and try to make sense of why they are what they are.

Here is another example of one-sided limits, similar to the last one:

Let $f : \mathbb{R}^* \to \mathbb{R}$ be defined by $f(x) = \frac{|x|}{x}$. What can we say about the left and right limits of $f(x)$ as x approaches 0? If x approaches 0 from the right, then x is positive, so $|x| = x$ and

$$f(x) = \frac{|x|}{x} = \frac{x}{x} = 1.$$

Hence no matter how close x is to 0 on the right, the value of $f(x)$ always remains 1. Therefore

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1.$$

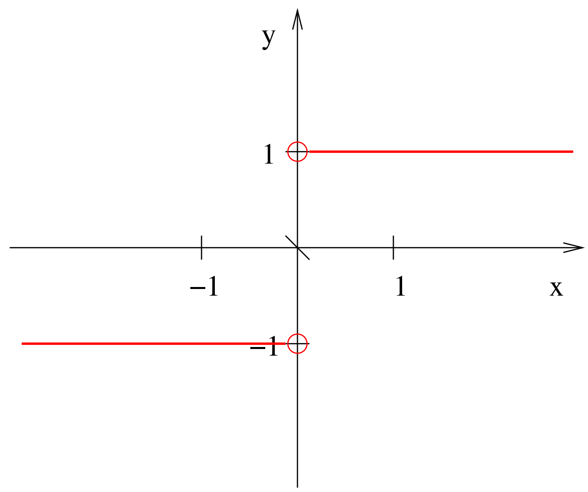
On the other hand, if x approaches 0 from the left, then x is negative, so $|x| = -x$ and

$$f(x) = \frac{|x|}{x} = \frac{-x}{x} = -1.$$

Hence no matter how close x is to 0 on the left, the value of $f(x)$ always remains -1. Therefore

$$\lim_{x \to 0^+} \frac{|x|}{x} = -1.$$

The values of the one-sided limits at 0 should also be quite clear from the graph:



Reading one-sided limits from the graph of a function is sometimes quite straightforward. This is what Example 7 in the textbook illustrates.

The relationship between one-sided limits and regular limits is as follows.

- If $\lim_{x \to a} f(x) = L$ then both $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a+} f(x) = L$. This should be quite clear from the informal definitions. If we can make the value of $f(x)$ as close as we want to L when x is sufficiently close to a, then the value of $f(x)$ will also be as close as we want to L when x is sufficiently close to a while $x > a$ and the value of $f(x)$ will also be as close as we want to L when x is sufficiently close to a while $x < a$. You can see a visual illustration of this as $x \to 5$ in Figure 10.
- If both $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a+} f(x) = L$ then $\lim_{x \to a} f(x) = L$. This should also be quite clear from the informal definitions. If we can make the value of $f(x)$ as close as we want to L when $x < a$ is sufficiently close to a and also when $x > a$ is sufficiently close to a, then the value of $f(x)$ will be as close as we want to L when x is sufficiently close to a regardless of whether x is to the left or to the right of a. What happens as $x \to 5$ in Figure 10 illustrates this idea as well.
- Finally, it is worth noting that if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a+} f(x)$ then $\lim_{x \to a} f(x)$ cannot exist because, as we have just noted, if $\lim_{x \to a} f(x)$ did exist, it would be some number L, but then both the left and the right limits would have to be equal to L, and hence they would be equal to each other.

  Note that $\lim_{x \to a^-} f(x) \neq \lim_{x \to a+} f(x)$ could be the case if the left and right limits are two different numbers, just like in the Heaviside function example above, or if one or both of the one-sided limits does not exist at all, just like in Example 8 in the textbook.

Example 8 is not really related to one-sided limits, it is there to remind you that the limit of a function at some point a may not exist at all, in this case because the value of $f(x) = 1/x^2$ gets larger and larger as $x \to 0$, and therefore cannot get close to any particular number.

Let us revisit the example

$$\lim_{xto2} x^2 = 4.$$

We noted that this means that we can make the value of $f(x) = x^2$ as close to 4 as we want by making x get sufficiently close to 2. To be more quantitative, we were able to make x^2 fall between

3.9 and 4.1 by making $1.99 < x < 2.01$, and then we made x^2 fall between 3.99 and 4.01 by making $1.999 < x < 2.001$. You may get the feeling that we can play this game as long as we want. If we really wanted, we could make x^2 fall between 3.99999999 and 4.00000001 by squeezing x between a number just a little less than 2 and a number just a little more than 2. But how do we really know we can do this? We can try to solve the following general problem: if ϵ is some (small) positive number, how close does x have to be to 2 so that x^2 is between $4-\epsilon$ and $4+\epsilon$? In other words, can we find a small enough positive number δ so that if

$$2-\delta < x < 2+\delta$$

then

$$4-\epsilon < x^2 < 4+\epsilon?$$

Let us give this a try. We want

$$4-\epsilon < x^2 < 4+\epsilon.$$

Can we solve this inequality (really two inequalities) for x? We can, but we need to do it carefully. It is tempting to take square roots of the three sides. But is that a legitimate step? In other words, do we know that if $a < b$ then $\sqrt{a} < \sqrt{b}$ for any numbers a and b? The answer is almost. First, a and b would have to be nonnegative numbers, so their square roots are real numbers. Indeed if $a < b$ are nonnegative numbers, then it is also true that $\sqrt{a} < \sqrt{b}$. This is because the square root function $g(x) = \sqrt{x}$ is an increasing function on its domain of $\mathbb{R}^{\geq 0}$. This means that if $a, b \in \mathbb{R}^{\geq 0}$ and $a < b$ then $g(a) < g(b)$, that is $\sqrt{a} < \sqrt{b}$. This is exactly what we said before. Now, we are interested in what happens when ϵ becomes small (but positive). If ϵ is small then $4-\epsilon$ is going to be nonnegative. So x will also be nonnegative. And for sure, $4+\epsilon$ is nonnegative since it is 4 plus a positive number. So in fact, if

$$4-\epsilon < x^2 < 4+\epsilon.$$

then

$$\sqrt{4-\epsilon} < \sqrt{x^2} < \sqrt{4+\epsilon}.$$

Now, $\sqrt{x^2}$ is usually $|x|$, but x in our case is a number close to 2, so $|x| = x$. Therefore

$$\sqrt{4-\epsilon} < x < \sqrt{4+\epsilon}.$$

We can try this with a specific number, say $\epsilon = 0.1$ to see how it works out. So for

$$4-0.1 < x^2 < 4+0.1$$

we would want

$$\sqrt{3.9} < x < \sqrt{4.1}.$$

If you want to see this with decimal approximations, say to four decimal digits:

$$1.9749 < x < 2.0248$$

At the risk of digressing a bit, it is worth discussing here how I came up with these decimal approximations. First,

$$\sqrt{3.9} = 1.9748417658\ldots$$

Using the usual rules of rounding, $\sqrt{3.9} \approx 1.9748$. But that is not the right number for our purposes here. Notice that if we chose x to be just a little more than 1.9748, say $x = 1.97481$, then x would actually be less than $\sqrt{3.9}$ and sure enough $1.97481^2 = 3.8998745361$ is not between 3.9 and 4.1. So the right thing to do is to always round the decimal approximation of the lower bound up. Similarly, we would always want to round the decimal approximation for the upper bound down. In fact,

$$\sqrt{4.1} = 2.0248456731\ldots,$$

and so we used 2.0248 as the decimal approximation above. Back to

$$1.9749 < x < 2.0248,$$

we could ask how close x actually has to be to 2. Since $1.9749 = 2-0.0251$ and $2.0248 = 2+0.0248$, we want

$$2 - 0.0251 < x < 2 + 0.0248$$

In other words, x needs to be closer to 2 than a distance of 0.0248 on the right and a distance of 0.0251 on the left. We can take the smaller of these two numbers and say that if x is less than a distance of 0.0248 from 2 (on the left or right, it does not matter), then x is close enough to 2 to be between

$$2 - 0.0248 < x < 2 + 0.0248$$

and hence we can be certain that

$$1.9749 = 2 - 0.0251 < x < 2 + 0.0248 = 2.0248$$

which is probably close enough to guarantee that x^2 is between 3.9 and 4.1. I said probably because we have not yet verified that it is actually true that if

$$1.9749 < x < 2.0248$$

then

$$3.9 < x^2 < 4.1.$$

This is easy enough to do. As we noted before, the function $f(x) = x^2$ is increasing on $\mathbb{R}^{\geq 0}$ and hence

$$1.9749 < x < 2.0248 \implies 1.9749^2 < x^2 < 2.0248^2 => 3.90023001 < x^2 < 4.09981504 \implies 3.9 < x^2 < 4.1,$$

which is what we wanted to know.

Here is the argument without decimal approximations. If

$$\sqrt{3.9} < x < \sqrt{4.1}$$

then

$$\sqrt{3.9}^2 < x^2 < \sqrt{4.1}^2$$

that is

$$3.9 < x^2 < 4.1$$

We can now do try the same thing with an arbitrary small positive number ϵ. We want

$$4 - \epsilon < x^2 < 4 + \epsilon$$

and as we argued above, this leads us to

$$\sqrt{4-\epsilon} < x < \sqrt{4+\epsilon}.$$

How close does x have to be to 2? It needs to be closer than a distance of $\sqrt{4+\epsilon} - 2$ on the right and a distance of $2 - \sqrt{4-\epsilon}$ on the left. It is important to note that both of these numbers are positive, since

$$4 + \epsilon > 4 \implies \sqrt{4+\epsilon} > \sqrt{4} = 2 \implies \sqrt{4+\epsilon} - 2 > 0$$

and

$$4 - \epsilon < 4 \implies \sqrt{4-\epsilon} < \sqrt{4} = 2 \implies 2 - \sqrt{4-\epsilon} > 0,$$

otherwise it would not be right to refer to them as distances. Once again, we can choose the smaller of these two numbers, that is let

$$\delta = \min(2 - \sqrt{4-\epsilon}, \sqrt{4+\epsilon} - 2)$$

Now suppose

$$2 - \delta < x < 2 + \delta$$

Note that

$$\delta \leq 2 - \sqrt{4-\epsilon} \quad \text{and} \quad \delta \leq \sqrt{4+\epsilon} - 2.$$

So

$$\delta \le 2 - \sqrt{4-\epsilon} \implies -\delta \ge \sqrt{4-\epsilon} - 2 \implies 2 - \delta \ge \sqrt{4-\epsilon} - 2 + 2 = \sqrt{4-\epsilon}$$

and

$$\delta \le \sqrt{4+\epsilon} - 2 \implies 2 + \delta \le \sqrt{4+\epsilon} - 2 + 2 = \sqrt{4+\epsilon}.$$

Therefore

$$\sqrt{4-\epsilon} \le 2 - \delta < x < 2 + \delta \le \sqrt{4+\epsilon}$$

and we can conclude

$$\sqrt{4-\epsilon} < x < \sqrt{4+\epsilon}.$$

All three sides of this inequality are nonnegative numbers, two of them because they are square roots and x is because it is between them. So we square all sides (remember $f(x) = x^2$ is increasing on $\mathbb{R}^{\ge 0}$):

$$\sqrt{4-\epsilon}^2 < x^2 < \sqrt{4+\epsilon}^2$$

and that says

$$4 - \epsilon < x^2 < 4 + \epsilon.$$

Success! We have been able to show that no matter how small a number $\epsilon > 0$ is, we can make x^2 fall between $4 - \epsilon$ and $4 + \epsilon$ as long as x is close enough to 2 to be satisfy

$$2 - \delta < x < 2 + \delta$$

where $\delta = \min(2 - \sqrt{4-\epsilon}, \sqrt{4+\epsilon} - 2)$. Note that it is not the actual value of δ that matters but that such a positive value exists.

This was an elaborate argument, but it has allowed us to make sense of what "arbitrarily close" and "sufficiently close" actually mean in terms of actual distances from 4 and 2. In fact, we can rephrase what we just did even more explicitly in terms of distances. Remember that the distance between two numbers x and y on the number line is $|x - y|$. Because of the absolute value, it does not matter we are subtracting the smaller number from the bigger number or the other way. So another way to say that x^2 is between $4 - \epsilon$ and $4 + \epsilon$ is to say that x^2 is closer to 4 than a distance of ϵ, or that

$$|x^2 - 4| < \epsilon.$$

By the same logic, making x sufficiently close to 2 to be between $2 - \delta$ and $2 + \delta$ is to say that x is closer to 2 than a distance of δ or

$$|x - 2| < \delta.$$

So what we showed is that for any positive number ϵ, however small, we can find a positive number δ such that if

$$|x - 2| < \delta$$

then

$$|x^2 - 4| < \epsilon.$$

To generalize this to other functions and their limits, we would say that

$$\lim_{x \to a} f(x) = L$$

means that for any positive number ϵ, however small, we can find a positive number δ such that if

$$|x - a| < \delta$$

then

$$|f(x) - L| < \epsilon.$$

That is almost the perfect way to say what a limit is. There is one more thing we need to be careful about. Remember that we can talk about the limit of a function $f(x)$ as x approaches a even if $f(a)$ does not have a value. See Figure 3 in Example 2 in your textbook. Or it could be that $f(a)$ exists but it is not the number that the value of $f(x)$ gets close to as x gets closer and closer to a but does not actually reach a. For this reason, we really do not want to consider what happens to $f(x)$ when $x = a$ or when $|x - a| = 0$. So the correct way to define a limit formally has $0 < |x - a| < \delta$ in it instead of just $|x - a| < \delta$. By saying $0 < |x - a| < \delta$, we are saying x is sufficiently close to a but not equal to a. Here is the complete defition.

Definition 1. Let f be a function of real numbers and let $a \in \mathbb{R}$ such that $f(x)$ is defined for every x in some neighborhood of a, except possibly at a. Then we say that the number L is the *limit of $f(x)$ as x approaches a* if for every $\epsilon > 0$ there exists a corresponding $\delta > 0$ such that if

$$0 < |x - a| < \delta$$

then

$$|f(x) - L| < \epsilon.$$

The notation is still the same:

$$\lim_{x \to a} f(x) = L.$$

This definition is a bit tricky to understand and work with, so let us look at some examples. First, look at Example 9 in your textbook for a very simple example. Second, notice that we have already done an example of a $\delta - \epsilon$ argument for $\lim_{x \to 2} x^2 = 4$ above. Let us revisit this example to give a slightly different argument involving absolute value inequalities, somewhat similar to how Example 9 is done in the textbook.

So we want to prove

$$\lim_{x \to 2} x^2 = 4.$$

In terms of the formal definition of the limit, we want to show that for any $\epsilon > 0$, however small, there is a corresponding $\delta > 0$ such that if

$$0 < |x - 2| < \delta$$

then

$$|x^2 - 4| < \epsilon.$$

So we want

$$|x^2 - 4| < \epsilon$$

to be true. First, we factor $x^2 - 4$ as

$$x^2 - 4 = (x + 2)(x - 2).$$

Now, note that $|ab| = |a||b|$ for any $a, b \in \mathbb{R}$. (Think a little about why this is true.) Therefore

$$|x^2 - 4| = |(x + 2)(x - 2)| = |x + 2||x - 2|.$$

So we want

$$|x + 2||x - 2| < \epsilon$$

Now, we can make $|x-2|$ small, as small as we need to, by making x be closer to 2 than a sufficiently small distance δ. We do not have as much control over $|x + 2|$. But if x is close to 2, then $x + 2$ will be a number close to 4, and so $|x + 2|$ is also a number close to 4. So

$$|x + 2||x - 2| \approx 4|x - 2|.$$

Roughly speaking, what we want to do is make $|x - 2| < \epsilon/4$. Then

$$|x^2 - 4| = |x + 2||x - 2| \approx 4|x - 2| < 4\,\frac{\epsilon}{4} = \epsilon.$$

This almost works except for the $\approx$ instead of $=$ in the middle. The problem is that while the value of $x + 2$ must be close to 4, it could be a little more than 4. And if it is then $|x + 2||x - 2|$ could end up being a little more than $4\,\frac{\epsilon}{4} = \epsilon$. But we are close to the truth here, we just need to make the argument more precise. While we cannot guarantee that $x+2=4$, it is easy to make sure

$$3 < x+2 < 5.$$

All we need for this is that

$$1 < x < 3,$$

which is not a lot to ask for if x is close to 2. In fact, another way to say this is that we need

$$|x-2| < 1.$$

So whatever value of δ we eventually settle on, we need to make sure it is no more than 1. We will need to remember this later. For now, let us say that if x is sufficiently close to 2 that

$$|x-2| < 1$$

then we know

$$1 < x < 3 \implies 3 < x+2 < 5 \implies 3 < |x+2| < 5 \implies 3|x-2| < |x+2||x-2| < 5|x-2|.$$

It is the last inequality that is particularly valuable for us. It tells us

$$|x^2-4| = |x+2||x-2| < 5|x-2|.$$

So if we make $|x-2| < \epsilon/5$ then we have

$$|x^2-4| = |x+2||x-2| < 5|x-2| < 5\,\frac{\epsilon}{5} = \epsilon.$$

What we want to do is set $\delta = \epsilon/5$. But wait, we also need to make sure that $\delta \leq 1$. So let us choose

$$\delta = \min\left(\frac{\epsilon}{5}, 1\right).$$

Notice that δ is definitely positive since $\epsilon/5$ and 1 are both positive. We also know that

$$\delta \leq \frac{\epsilon}{5}$$

and

$$\delta \leq 1.$$

We will now show that if

$$0 < |x-2| < \delta$$

then

$$|x^2-4| < \epsilon.$$

So suppose

$$0 < |x-2| < \delta.$$

This tells us that

$$|x-2| < \delta \leq \frac{\epsilon}{5}$$

and also

$$|x-2| < \delta \leq 1.$$

From the latter,

$$|x-2| < 1 \implies 1 < x < 3 \implies 3 < x+2 < 5 \implies 3 < |x+2| < 5$$

Now

$$|x^2-4| = \underbrace{|x+2|}_{<5}\,\underbrace{|x-2|}_{<\epsilon/5} < 5\,\frac{\epsilon}{5} = \epsilon,$$

which is what we wanted to show.

Before we look at another example, you should take a careful look at Figures 12-14 in the textbook. These illustrate what the formal definition of the limit says. You can see this in action if you play with the Desmos app about the δ and ϵ definition of the limit, which illustrates the same ideas in a dynamic and interactive manner.

As another example, let us consider

$$\lim_{x\to 5} \frac{1}{x}.$$

We can reason that if x is a number that is close to 5 then $1/x$ should be close to $1/5$, and the closer x gets to 5, the closer $1/x$ should be to $1/5$. You can see this in the Desmos δ and ϵ app too. We will now show this using the formal definition of the limit.

We need to show that for any $\epsilon > 0$, however small, there is a corresponding $\delta > 0$ such that if

$$0 < |x - 5| < \delta$$

then

$$\left|\frac{1}{x} - \frac{1}{5}\right| < \epsilon.$$

So we want

$$\left|\frac{1}{x} - \frac{1}{5}\right| < \epsilon.$$

This is equivalent to

$$\frac{1}{5} - \epsilon < \frac{1}{x} < \frac{1}{5} + \epsilon.$$

We can solve these two inequalities for x. We will start with the one on the right:

$$\frac{1}{x} < \frac{1}{5} + \epsilon.$$

We can multiply both sides by x, but we need to be careful when multiplying an inequality by a number because the direction of the inequality could reverse if the number is negative. In this case, x should be a number close to 5, so x should be positive. In fact, we can easily make sure

$$4 < x < 6$$

by setting

$$|x - 5| < 1.$$

So in the end, whatever value we settle on for δ, we will make sure it is not more than 1. With that concern out of the way, let us multiply both sides of the inequality by x, which we know is positive:

$$\begin{aligned} x\,\frac{1}{x} &< x\left(\frac{1}{5} + \epsilon\right) \\ 1 &< x\left(\frac{1}{5} + \epsilon\right) \end{aligned}$$

We will now want to divide both sides by $1/5 + \epsilon$. Once again, we need to be careful to think about the sign of the number we are dividing by. We know $\epsilon > 0$, so $1/5 + \epsilon$ is for sure positive. Hence

$$\frac{1}{\frac{1}{5} + \epsilon} < x$$

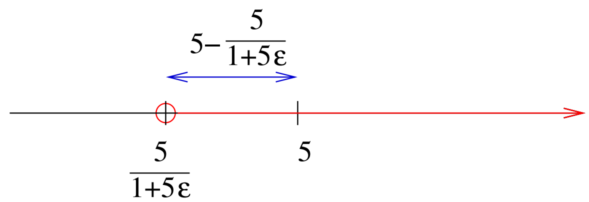
We can simplify the fraction on the left-hand side somewhat:

$$\frac{1}{\frac{1}{5} + \epsilon} = \frac{5}{5\left(\frac{1}{5} + \epsilon\right)} = \frac{5}{1 + 5\epsilon}.$$

We now have

$$\frac{5}{1 + 5\epsilon} < x.$$

As a bit of reality check, let us think about what the value of $\frac{5}{1+5\epsilon}$ might be. We know $\epsilon > 0$. In fact, we can think of ϵ as a small positive number. So 5ϵ is also a small positive number and $1+5\epsilon$ is a little more than 1. Therefore $\frac{5}{1+5\epsilon}$ is going to be a little less than 5. That is what we would expect: it says that x should not be too far from x on the left:



In this case, we would want to set δ to be the distance between $\frac{5\epsilon}{1+5\epsilon}$ and 5, that is

$$\delta = 5 - \frac{5\epsilon}{1+5\epsilon} = \frac{5(1+5\epsilon)}{1+5\epsilon} - \frac{5}{1+5\epsilon} = \frac{5+25\epsilon-5}{1+5\epsilon} = \frac{25\epsilon}{1+5\epsilon}.$$

We should not forget that we also said δ should not be more than 1. But we can deal with that later.

Now, we can tackle the other inequality similarly. We can multiply

$$\frac{1}{5} - \epsilon < \frac{1}{x}$$

by x since we already noted that x is positive:

$$x\left(\frac{1}{5} - \epsilon\right) < x\,\frac{1}{x}$$

$$x\left(\frac{1}{5} - \epsilon\right) < 1$$

We now want to divide by $1/5 - \epsilon$, but this number could be negative or even 0 if $\epsilon \geq 1/5$. But in that case, the inequality is already satisfied since we already know x is positive and hence $1/x$ is positive. So if $\epsilon \geq 1/5$ no upper bound on x needed to guarantee that

$$\frac{1}{5} - \epsilon < \frac{1}{x}.$$

Therefore setting

$$\delta = \frac{25\epsilon}{1+5\epsilon},$$

as we did above should be sufficiently small for our needs.

If $\epsilon < 1/5$ then $1/5 - \epsilon$ is positive. We will divide by it.

$$x\left(\frac{1}{5} - \epsilon\right) < 1$$

$$x < \frac{1}{\frac{1}{5} - \epsilon}$$

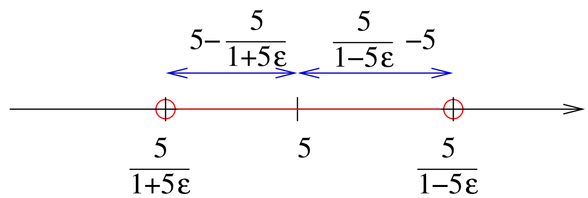
We can simplify the fraction on the right-hand side somewhat:

$$\frac{1}{\frac{1}{5} - \epsilon} = \frac{5}{5\left(\frac{1}{5} - \epsilon\right)} = \frac{5}{1-5\epsilon}.$$

We now have

$$x < \frac{5}{1-5\epsilon}.$$

For a bit of reality check, note that if ϵ is a small positive number then 5ϵ is also a small positive number and $1 - 5\epsilon$ is a little less than 1. Therefore $\frac{5}{1-5\epsilon}$ is going to be a little more than 5. That is what we would expect: it says that x should not be too far from x on the right. Here is what we have so far:



How far can x be from 5? It should not be so far that it would be to the left of $\frac{5}{1+5\epsilon}$ or to the right of $\frac{5}{1-5\epsilon}$. So it should be closer to 5 than whichever of these two numbers is closer to 5 on the number line. That is we want to choose δ to be the smaller of $\frac{5}{1-5\epsilon} - 5$ and $5 - \frac{5}{1+5\epsilon}$. We can simplify these expressions a bit:

$$\frac{5}{1-5\epsilon} - 5 = \frac{5}{1-5\epsilon} - \frac{5(1-5\epsilon)}{1-5\epsilon} = \frac{5-5+25\epsilon}{1-5\epsilon} = \frac{25\epsilon}{1-5\epsilon}$$

and as we already showed above,

$$5 - \frac{5}{1+5\epsilon} = \frac{25\epsilon}{1+5\epsilon}.$$

We could just set δ equal to the minimum of these two numbers. In fact, it is easy enough to see that the smaller number will always be $\frac{25\epsilon}{1+5\epsilon}$. This is because both $\frac{25\epsilon}{1+5\epsilon}$ and $\frac{25\epsilon}{1-5\epsilon}$ have the same positive numerator, the denominators are also positive, and $1+5\epsilon$ is the larger of the two denominators. So

$$\min\left(\frac{25\epsilon}{1-5\epsilon}, \frac{25\epsilon}{1+5\epsilon}\right) = \frac{25\epsilon}{1+5\epsilon}.$$

Therefore we would want to set

$$\delta = \frac{25\epsilon}{1+5\epsilon},$$

just like in the case $\epsilon \geq 1/5$. We should not forget that we also said δ should not be more than 1. So we will in fact set

$$\delta = \min\left(\frac{25\epsilon}{1+5\epsilon}, 1\right).$$

We need to be sure this is a positive number. Since $\epsilon > 0$, both the numerator and the denominator of $\frac{25\epsilon}{1+5\epsilon}$ are positive. And of course, 1 is a positive number. So whichever of these two numbers is smaller is still going to be positive.

Now we need to show that if

$$0 < |x-5| < \delta$$

then

$$\left|\frac{1}{x} - \frac{1}{5}\right| < \epsilon.$$

So suppose

$$0 < |x-5| < \delta.$$

Then

$$5 - \delta < x < 5 + \delta.$$

We want to show that

$$\frac{1}{5} - \epsilon < \frac{1}{x} < \frac{1}{5} + \epsilon.$$

We start with

$$5 - \delta < x.$$

Since $\delta \le 1$, we know $4 \le 5-\delta < x$, and in particular, both $5-\delta$ and x are positive. Therefore

$$5-\delta < x \implies 1 < \frac{x}{5-\delta} \implies \frac{1}{x} < \frac{1}{5-\delta}.$$

Since

$$\delta = \min\left(\frac{25\epsilon}{1+5\epsilon}, 1\right),$$

we know that

$$\delta \le \frac{25\epsilon}{1+5\epsilon}.$$

Therefore

$$\delta < \frac{25\epsilon}{1+5\epsilon} \implies -\delta > -\frac{25\epsilon}{1+5\epsilon} \implies 5-\delta > 5-\frac{25\epsilon}{1+5\epsilon} = \frac{5(1+5\epsilon)}{1+5\epsilon} - \frac{25\epsilon}{1+5\epsilon} = \frac{5+25\epsilon-25\epsilon}{1+5\epsilon} = \frac{5}{1+5\epsilon}.$$

Since $\frac{5}{1+5\epsilon}$ and $5-\delta$ are both positive,

$$\frac{5}{1+5\epsilon} < 5-\delta \implies \frac{1}{5-\delta}\frac{5}{1+5\epsilon} < 1 \implies \frac{1}{5-\delta} < \frac{1}{\frac{5}{1+5\epsilon}} = \frac{1+5\epsilon}{5} = \frac{1}{5}+\epsilon.$$

We can now conclude

$$\frac{1}{x} < \frac{1}{5-\delta} < \frac{1}{5}+\epsilon.$$

We will now prove $1/5-\epsilon < 1/x$. First, note that if $\epsilon \ge 1/5$ then $1/5-\epsilon \le 0$, and since x is positive,

$$\frac{1}{5}-\epsilon < \frac{1}{x}$$

definitely holds. If $\epsilon < 1/5$, we will use the other inequality:

$$x < 5+\delta.$$

Since x and $5+\delta$ are both positive,

$$x < 5+\delta \implies \frac{x}{5+\delta} < 1 \implies \frac{1}{5+\delta} < \frac{1}{x}.$$

Because

$$\delta = \min\left(\frac{25\epsilon}{1+5\epsilon}, 1\right),$$

we know that

$$\delta \le \frac{25\epsilon}{1+5\epsilon}.$$

Remember that we already argued that

$$\frac{25\epsilon}{1+5\epsilon} < \frac{25\epsilon}{1-5\epsilon}.$$

because both fractions have the same positive numerator, positive denominators, and the fraction on the left-hand side has larger denominator. Therefore

$$\delta < \frac{25\epsilon}{1-5\epsilon} \implies 5+\delta \le 5+\frac{25\epsilon}{1-5\epsilon} = \frac{5(1-5\epsilon)}{1-5\epsilon} + \frac{25\epsilon}{1-5\epsilon} = \frac{5-25\epsilon+25\epsilon}{1-5\epsilon} = \frac{5}{1-5\epsilon}.$$

Since $5+\delta < \frac{5}{1+5\epsilon}$ are both positive numbers,

$$5+\delta < \frac{5}{1+5\epsilon} \implies 1 < \frac{1}{5+\delta}\frac{5}{1+5\epsilon} \implies \frac{1}{5+\delta} > \frac{1}{\frac{5}{1-5\epsilon}} = \frac{1-5\epsilon}{5} = \frac{1}{5}-\epsilon,$$

and hence

$$\frac{1}{5}-\epsilon < \frac{1}{5+\delta} < \frac{1}{x}.$$

We can now conclude that

$$\frac{1}{5} - \epsilon < \frac{1}{x} < \frac{1}{5} + \epsilon \implies \left|\frac{1}{x} - \frac{1}{5}\right| < \epsilon.$$

This was quite a long and elaborate argument. There is an easier way to argue using the same idea we learned in the $\lim_{x\to 2} x^2 = 4$ example. We still want to show that for any $\epsilon > 0$, however small, there is a corresponding $\delta > 0$ such that if

$$0 < |x-5| < \delta$$

then

$$\left|\frac{1}{x} - \frac{1}{5}\right| < \epsilon.$$

So we want

$$\left|\frac{1}{x} - \frac{1}{5}\right| < \epsilon.$$

We start off by manipulating the expression in the absolute value:

$$\frac{1}{x} - \frac{1}{5} = \frac{5}{5x} - \frac{x}{5x} = \frac{5-x}{5x}.$$

We will now use the property $|a/b| = |a|/|b|$ for any $a, b \in \mathbb{R}$ such that $b \neq 0$. (Think about why this is true.)

$$\left|\frac{1}{x} - \frac{1}{5}\right| = \left|\frac{5-x}{5x}\right| = \frac{|5-x|}{|5x|}.$$

Note that $|5-x| = |x-5|$. This is because $5 - x = -(x-5)$ so the only difference between the numbers $5-x$ and $x-5$ is their sign. One of them is positive and the other is negative (or they are both 0), and the absolute value does not really care about signs. Another way to think about this is that both $|5-x|$ and $|x-5|$ are the distance between x and 5 on the number line. So

$$\left|\frac{1}{x} - \frac{1}{5}\right| = \frac{|x-5|}{|5x|}.$$

We can make the numerator $|x-5|$ as small as we need to do. We do not have the same control over the denominator, but we know that if x is close to 5 then $5x$ is close to 25. So $|5x|$ is also close to 25. Hence $|5x| \approx 25$ and

$$\frac{|x-5|}{|5x|} \approx \frac{|x-5|}{25}.$$

Roughly speaking, what we want to do is make $|x-5| < 25\epsilon$. Then

$$\left|\frac{1}{x} - \frac{1}{5}\right| = \frac{|x-5|}{|5x|} \approx \frac{|x-5|}{25} < \frac{25\epsilon}{25} = \epsilon.$$

This almost works except that while the value of $5x$ must be close to 25, it could be a little less than 25. And if it is, then $\frac{|x-5|}{|5x|}$ could end up being a little more than $\frac{25\epsilon}{25} = \epsilon$.

We need to make the argument more precise. While we cannot guarantee that $5x = 25$, we can make sure

$$4 < x < 6$$

by setting

$$|x-5| < 1.$$

Hence

$$20 < 5x < 30 \implies 20 < |5x| < 30 \implies \frac{1}{30} < \frac{1}{|5x|} < \frac{1}{20}.$$

Therefore

$$\frac{|x-5|}{|5x|} = \frac{1}{|5x|}\,|x-5| < \frac{1}{20}\,|x-5|$$

Now, if we make $|x-5| < 20\epsilon$ then

$$\frac{|x-5|}{25} < \frac{1}{20}|x-5| = \frac{1}{20}20\epsilon = \epsilon.$$

Hence we will want to set $\delta = 20\epsilon$. But remember that we also want

$$|x-5| < 1,$$

so we do not want δ to be more than 1. Therefore we will actually set

$$\delta = \min(20\epsilon, 1).$$

We now need to show that if

$$0 < |x-5| < \delta$$

then

$$\left|\frac{1}{x} - \frac{1}{5}\right| < \epsilon,$$

by putting together the pieces. So suppose

$$0 < |x-5| < \delta.$$

Then we know

$$|x-5| < \delta \leq 20\epsilon$$

and

$$|x-5| < \delta \leq 1 \implies 4 < x < 6 \implies 20 < 5x < 30 \implies 20 < |5x| < 30 \implies \frac{1}{30} < \frac{1}{|5x|} < \frac{1}{20}.$$

Therefore

$$\left|\frac{1}{x} - \frac{1}{5}\right| = \frac{|x-5|}{|5x|} = \frac{1}{|5x|}|x-5| < \frac{1}{20}20\epsilon = \epsilon,$$

which is what we wanted to show.

This second argument is considerably shorter and more straightforward. Still, it is not exactly simple. There is no way around this: $\delta - \epsilon$ arguments are delicate and require good algebra skills.

Here is one more example to help you understand why we have the greater than 0 part in the condition $0 < |x-a| < \delta$. Consider

$$\lim_{x\to 1} \frac{x^2-1}{x-1}.$$

If x is close to 1, then x^2 should be close to 1 as well and x^2-1 will be close to 0. But $x-1$ should also be close to 0. It is hard to tell what happens when you divide a number close to 0 by another number close to 0. The result could be positive or negative, small (close to 0), or large, or medium in size, depending on the relative size of the numerator to the denominator. Notice that

$$\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1}.$$

It is tempting to cancel the factor $x-1$. That is fine to do if $x-1 \neq 0$:

$$\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x+1 \quad \text{if } x-1 \neq 0.$$

Of course, $x-1 \neq 0$ if and only if $x \neq 1$. In fact, if $x = 1$ then

$$\frac{x^2-1}{x-1} = \frac{1^2-1}{1-1} = \frac{0}{0}$$

is undefined, and is certainly not equal to $1+1=2$. But for any value of x close to 1, except $x=1$,

$$\frac{x^2-1}{x-1} = x+1$$

will be close to 2. In fact, the closer x gets to 1, the closer $x+1$ and hence $\frac{x^2-1}{x-1}$ should be to 2. Therefore it is reasonable to guess that

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2.$$

We will prove this using the formal definition of the limit. We need to show that for any $\epsilon > 0$ there is a corresponding $\delta > 0$ such that if

$$0 < |x-1| < \delta$$

then

$$\left|\frac{x^2-1}{x-1} - 2\right| < \epsilon.$$

So we want

$$\left|\frac{x^2-1}{x-1} - 2\right| < \epsilon.$$

We are only interested in what happens when $0 < |x-1| < \delta$ for some appropriately small value of δ. This tells us that x cannot be 1, otherwise $|x-1| = 0$. So for any value of x under consideration,

$$\frac{x^2-1}{x-1} = x+1 \implies \left|\frac{x^2-1}{x-1} - 2\right| = |x+1-2| = |x-1|.$$

The point is that as far as the limit is concerned, it does not matter what the value of $\frac{x^2-1}{x-1}$ is at exactly $x = 1$. What matters is the value of $\frac{x^2-1}{x-1}$ when x is close to 1 but not equal to 1. So we want

$$|x-1| < \epsilon.$$

But $|x-1|$ is exactly what we can make as small as we need to by choosing the value of δ wisely. It is quite clear that we want to set $\delta = \epsilon$ in this case. Since $\epsilon > 0$, δ is also positive.

Now, we will show that if

$$0 < |x-1| < \delta$$

then

$$\left|\frac{x^2-1}{x-1} - 2\right| < \epsilon.$$

So suppose

$$0 < |x-1| < \delta.$$

Since $\delta = \epsilon$, this tells us

$$0 < |x-1| < \epsilon.$$

Now,

$$\left|\frac{x^2-1}{x-1} - 2\right| = \left|\frac{(x+1)(x-1)}{x-1} - 2\right|.$$

Since $0 < |x-1|$, we know $x \neq 1$ and hence

$$\frac{(x+1)(x-1)}{x-1} = x+1.$$

So

$$\left|\frac{x^2-1}{x-1} - 2\right| = |x+1-2| = |x-1| < \epsilon,$$

which is what we wanted to show.

Notice that in the argument above, we actually needed to use that $0 < |x-1|$. In fact, if you look carefully, we have just argued that

$$\lim_{x\to 1} \frac{x^2-1}{x-1}$$

must be the same as

$$\lim_{x \to 1} (x+1).$$

In general, we could have made the same argument about the limits of two functions $f(x)$ and $g(x)$ as $x \to a$ if $f(x) = g(x)$ for every $x \neq a$ to conclude

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x).$$

We will see that this idea is often helpful in finding the limit of a more complicated function by replacing it with a simpler function.

Let us look at some examples when the limit does not exist. We have seen that

$$\lim_{x \to 0} \frac{|x|}{x}$$

does not exist because the left and the right limits are different. But we can learn from taking a closer look at how the $\delta - \epsilon$ definition fails to be satisfied. For example, we can show that whatever number L is, for $\epsilon = 1$ there cannot exist a $\delta > 0$ such that if

$$0 < |x - 0| < \delta$$

then

$$\left| \frac{|x|}{x} - L \right| < \epsilon.$$

Let $f(x) = |x|/x$. Remember that $f(x) = -1$ if x is negative and $f(x) = 1$ if x is positive. We want

$$\left| \frac{|x|}{x} - L \right| < 1,$$

that is

$$L - 1 < \frac{|x|}{x} < L + 1$$

whenever

$$-\delta < x < \delta \quad \text{and} \quad x \neq 0.$$

But no matter how small we make $\delta > 0$, there will always be both positive and negative values of x that satisfy

$$-\delta < x < \delta \quad \text{and} \quad x \neq 0.$$

For example, $x_1 = -\delta/2$ is a negative number that satisfies

$$-\delta < x < \delta \quad \text{and} \quad x \neq 0,$$

and $x_2 = \delta/2$ is a positive number that does. Now, $f(x_1) = -1$ and $f(x_2) = 1$, and -1 and 1 are two units apart on the number line, so they cannot both fit inside the open interval $(L-1, L+1)$. Here is a more algebraic argument to show this. If x_1 satisfied

$$L - 1 < \frac{|x_1|}{x_1} < L + 1$$

then we would have

$$L - 1 < -1 < L + 1 \implies L - 1 < -1 \implies L < 0.$$

If x_2 satisfied

$$L - 1 < \frac{|x_2|}{x_2} < L + 1$$

then we would have

$$L - 1 < 1 < L + 1 \implies 1 < L + 1 \implies 0 < L.$$

But L cannot be both less than and greater than 0.

We have argued at least informally that

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$$

does not exist. We can give a similar argument for why it is not possible for any real number L to satisfy the formal definition of the limit. Once again, I will show that if $\epsilon = 1$ there is no corresponding $\delta > 0$ such that if

$$0 < |x - 0| < \delta$$

then

$$\left|\sin\left(\frac{\pi}{x}\right) - L\right| < \epsilon.$$

Let $f(x) = \sin\left(\frac{\pi}{x}\right)$.

$$|f(x) - L| < 1,$$

that is

$$L - 1 < f(x) < L + 1$$

whenever

$$-\delta < x < \delta \quad \text{and} \quad x \neq 0.$$

But no matter how small we make $\delta > 0$, there will always be some values of x that satisfy

$$-\delta < x < \delta \quad \text{and} \quad x \neq 0,$$

for which $f(x) = 1$ and other values of x for which $f(x) = -1$. Here is why. Let n be an integer number such that $n > 1/\delta$. Remember that δ can be quite small, but is always positive, so $1/\delta$ is some positive number, possibly quite large. But it is always possible to find an integer that is bigger than $1/\delta$. Now, set

$$x_1 = \frac{1}{2n + 1/2} \quad \text{and} \quad x_2 = -x_1 = -\frac{1}{2n + 1/2}.$$

Note that

$$\frac{1}{\delta} < n < 2n + \frac{1}{2} \implies 1 < \delta\left(2n + \frac{1}{2}\right) \implies \frac{1}{2n + 1/2} < \delta \implies 0 < x_1 < \delta.$$

Therefore it is also true that

$$0 > -x_1 > -\delta \implies -\delta < x_2 < 0.$$

So both x_1 and x_2 are closer to 0 than a distance of δ. Now

$$f(x_1) = \sin\left(\frac{\pi}{x_1}\right) = \sin\left(\frac{\pi}{\frac{1}{2n+1/2}}\right) = \sin\left(2n\pi + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$f(x_2) = f(-x_1) = \sin\left(\frac{\pi}{-x_1}\right) = \sin\left(-2n\pi - \frac{\pi}{2}\right) = \sin\left(-\frac{\pi}{2}\right) = -1$$

By the same argument as in the previous example, $f(x_1) = 1$ and $f(x_2) = -1$ cannot both fit inside the open interval $(L - 1, L + 1)$. Therefore no matter how small we make δ, we cannot have

$$|f(x) - L| < 1$$

for every x such that

$$0 < |x - 0| < \delta.$$

Now that we have developed some understanding of the formal definition of the limit, we can construct similar formal definitions for one-sided limits. We may say that

$$\lim_{x\to a^+} f(x) = L$$

if for every $\epsilon > 0$ there exists a corresponding $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

whenever $x > a$ is such that

$$0 < |x - a| < \delta$$

This is fine, but unnecessarily complicated. In fact, notice that

$$0 < |x - a| < \delta$$

is equivalent to

$$a - \delta < x < a + \delta \quad \text{and} \quad x \neq a.$$

But we also have $x > a$, so x must satisfy $a < x < a + \delta$. Here is the complete definition

Definition 2. Let f be a function of real numbers and let $a \in \mathbb{R}$ such that $f(x)$ is defined for every x in some neighborhood $(a, a + \delta)$. Then we say

$$\lim_{x \to a^+} f(x) = L$$

if for every $\epsilon > 0$ there exists a corresponding $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

whenever

$$a < x < a + \delta.$$

We can define the left limit similarly:

Definition 3. Let f be a function of real numbers and let $a \in \mathbb{R}$ such that $f(x)$ is defined for every x in some neighborhood $(a, a + \delta)$. Then we say

$$\lim_{x \to a^-} f(x) = L$$

if for every $\epsilon > 0$ there exists a corresponding $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

whenever

$$a - \delta < x < a.$$

You can use these definitions very similarly to the formal definition of the two-sided limit. For example, let

$$f(x) = \begin{cases} -1 & \text{if } x < -2 \\ x^2 & \text{if } x \geq -2 \end{cases}$$

and let us look at the one-sided limits:

$$\lim_{x \to -2^-} f(x) \quad \text{and} \quad \lim_{x \to -2^+} f(x)$$

First, it is easy enough to see that the left limit as $x \to -2^-$ should be -1 since the value of $f(x)$ is always -1 for any $x < -2$. In fact, it is really easy to show that for every $\epsilon > 0$ there exists a corresponding $\delta > 0$ such that if

$$-2 - \delta < x < -2.$$

then

$$|f(x) - (-1)| < \epsilon.$$

For any $\epsilon > 0$, we can choose δ to be any positive number we like. Say $\delta = 1$. If

$$-2 - \delta < x < -2$$

then

$$-3 < x < -2 \implies f(x) = -1 \implies |f(x) - (-1)| = |-1 - (-1)| = 0 < \epsilon.$$

The right limit takes a slightly more elaborate argument, but it is almost the same as the one we used to show $\lim_{x\to 2} x^2 = 4$. Now, if x is close to -2 on the right, then $x > -2$ and hence $f(x) = x^2$, which is close to 4. Therefore it is reasonable to guess that

$$\lim_{x\to -2^+} f(x) = 4.$$

We need to show that for every $\epsilon > 0$ there exists a corresponding $\delta > 0$ such that if

$$-2 < x < -2 + \delta.$$

then

$$|f(x) - 4| < \epsilon.$$

First, since $-2 < x$, $f(x) = x^2$, and so

$$|f(x) - 4| < |x^2 - 4|.$$

We want this to be less than ϵ. We will proceed as before by factoring

$$x^2 - 4 = (x + 2)(x - 2),$$

and hence

$$|x^2 - 4| = |(x + 2)(x - 2)| = |x + 2||x - 2|.$$

In this case, it is $|x + 2|$ that we can make as small as we need to, since x is close to -2. And $x - 2$ will be close to -4 and therefore $|x - 2| \approx 4$. Similarly to how we handled this before, we will require that

$$-2 < x < -1$$

or in other words

$$-2 < x < -2 + 1$$

So we need to make sure that our final choice for δ satisfies $\delta \leq 1$. Now

$$-2 < x < -1 \implies -4 < x - 2 < -3 \implies 3 < |x - 2| < 4 \implies 3|x + 2| < |x - 2||x + 2| < 4|x + 2|.$$

Hence making $|x + 2| < \epsilon/4$ will guarantee that

$$|x^2 - 4| = |x - 2||x + 2| < 4|x + 2| < 4\frac{\epsilon}{4} = \epsilon.$$

So let us set $\delta = \min(\epsilon/4, 1)$. Note that this guarantees $\delta \leq \epsilon/4$ and $\delta \leq 1$. Also, δ is positive because $\epsilon/4$ and 1 are positive.

We need to show that if

$$-2 < x < -2 + \delta.$$

then

$$|f(x) - 4| < \epsilon.$$

So let

$$-2 < x < -2 + \delta.$$

Since $x > -2$, $f(x) = x^2$ and

$$|f(x) - 4| = |x^2 - 4| = |(x + 2)(x - 2)| = |x + 2||x - 2|.$$

We know

$$-2 < x < -2 + \delta \implies 0 < x + 2 < \delta \implies 0 < |x + 2| < \delta \leq \epsilon/4$$

and

$$-2 < x < -2 + \delta \implies -4 < x - 2 < -4 + \delta \leq -4 + 1 = -3 \implies 3 < |x - 2| < 4.$$

Therefore

$$|f(x) - 4| = |x + 2||x - 2| < 4\frac{\epsilon}{4} = \epsilon.$$

See Example 10 in your textbook for another one-sided limit argument.

Finally, note that the formal definition does not help you calculate a limit. It is a theoretical tool that is used to prove that the limit of a function is in fact the number we guessed some other way. The guess may come from a table of values, a graph, or just by thinking about how the output values of the function should behave when the input value approaches a number a. We have seen these last three are imperfect computational tools and they can easily lead us to the wrong guess. The formal $\delta - \epsilon$ argument is what allows us to have confidence that the guess we derived from an imperfect tool is indeed correct.

Guessing has its pitfalls and constructing $\delta - \epsilon$ arguments is both challenging and time-consuming. In the next section, we will develop some tools to allow us to determine limits quickly and easily, without imperfect guessing and without having to constructing elaborate $\delta - \epsilon$ arguments.