# Notes for Section 1.4

Now that we have seen how to deal with limits using the definition (hard), we will learn how to find them using a few shortcuts (easy). The shortcuts do not always work, but they do work often. The first thing you will want to learn is the limit laws. These are of course theorems, each of which can be proven to be true either by a $\delta - \epsilon$ argument of by using the other limits laws. We will show why some of these are true, but first focus on what they say and how to use them. I will not list them here since they are listed in the textbook anyway. But let me show you how to use them on an example.

Let us find

$$\lim_{x\to 5} \frac{3x+4}{\sqrt{x^2-9}}.$$

I will do this by starting with the basic ingredients this function is made up of. Let us first look at the parts of the numerator since that is simpler than the denominator. By LL8,

$$\lim_{x\to 5} x = 5.$$

By LL3,

$$\lim_{x\to 5}(3x) = 3\lim_{x\to 5} x = 15.$$

Now, by LL7,

$$\lim_{x\to 5} 4 = 4.$$

And by LL1,

$$\lim_{x\to 5}(3x+4) = \lim_{x\to 5}(3x) + \lim_{x\to 5} 4 = 15 + 4 = 19.$$

On to the denominator. First,

$$\lim_{x\to 5} x^2 = 25$$

by LL9. By LL7,

$$\lim_{x\to 5} 9 = 9.$$

By LL2,

$$\lim_{x\to 5}(x^2-9) = \lim_{x\to 5} x^2 - \lim_{x\to 5} 9 = 25 - 9 = 16.$$

By LL11,

$$\lim_{x\to 5}\sqrt{x^2-9} = \sqrt{\lim_{x\to 5}(x^2-9)} = \sqrt{16} = 4.$$

Finally, by LL5,

$$\lim_{x\to 5} \frac{3x+4}{\sqrt{x^2-9}} = \frac{\lim_{x\to 5}(3x+4)}{\lim_{x\to 5}\sqrt{x^2-9}} = \frac{19}{4}.$$

This is a bit tedious, but also quite straightforward. It allows us to compute the limit and be certain that it is the number we computed. We did this by building up a more complicated function from its basic components from the bottom to the top. It is also possible to go the other way, top to bottom. The logic is on shakier grounds but the calculation is typically shorter. Here is how this limit could be found that way. We start with

$$\lim_{x\to 5} \frac{3x+4}{\sqrt{x^2-9}} = \frac{\lim_{x\to 5}(3x+4)}{\lim_{x\to 5}\sqrt{x^2-9}}$$

by LL5. If you have been paying attention, you know that LL5 can only be used if

$$\lim_{x\to 5}(3x+4) \quad \text{and} \quad \lim_{x\to 5}\sqrt{x^2-9}$$

both exist and the one in the denominator is not 0. We really do not know that yet. That is why I said the logic is on shakier grounds. We can resolve the issue by saying we can charge ahead and we will find out if the two limits exist and if the one in the denominator is 0. And if they do turn out to exist and the denominator turns out not to be 0 then we were justified in using LL5. So let us set doubts aside and proceed this way.

$$
\begin{aligned}
\lim_{x\to 5}\frac{3x+4}{\sqrt{x^2-9}} &= \frac{\lim_{x\to 5}(3x+4)}{\lim_{x\to 5}\sqrt{x^2-9}} && \text{by LL5}\\
&= \frac{\lim_{x\to 5}(3x+4)}{\sqrt{\lim_{x\to 5}(x^2-9)}} && \text{by LL11}\\
&= \frac{\lim_{x\to 5}(3x)+\lim_{x\to 5}4}{\sqrt{\lim_{x\to 5}x^2-\lim_{x\to 5}9}} && \text{by LL1 and LL2}\\
&= \frac{3\lim_{x\to 5}x+\lim_{x\to 5}4}{\sqrt{\lim_{x\to 5}x^2-\lim_{x\to 5}9}} && \text{by LL3}\\
&= \frac{3\lim_{x\to 5}x+\lim_{x\to 5}4}{\sqrt{5^2-\lim_{x\to 5}9}} && \text{by LL9}\\
&= \frac{3(5)+\lim_{x\to 5}4}{\sqrt{5^2-\lim_{x\to 5}9}} && \text{by LL8}\\
&= \frac{3(5)+4)}{\sqrt{5^2-9}} && \text{by LL7}\\
&= \frac{19}{\sqrt{16}}\\
&= \frac{19}{4}.
\end{aligned}
$$

This is the approach the textbook takes in Example 1.

Notice that if you substitute $x = 5$ into

$$f(x) = \frac{3x+4}{\sqrt{x^2-9}}$$

you also get $f(5) = 19/4$. But simply substituting $a$ into $x$ to calculate

$$\lim_{x\to a} f(x)$$

is not a safe thing to do. It does sometimes give the correct limit, as it did in this case, but in other cases it may not. For example, substituting $x = a$ into the function does not allow you to find the following limits:

$$
\begin{gathered}
\lim_{x\to 0} H(x)\\
\lim_{x\to 0}\frac{|x|}{x}\\
\lim_{x\to 0}\sin\left(\frac{\pi}{x}\right)\\
\lim_{x\to 1}\frac{x^2-1}{x-1}\\
\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^3}
\end{gathered}
$$

where $H(x)$ is the Heaviside function from Example 6 in Section 1.3. It would be good to at least know when we can use simple subsitution to find the limit of a function. But that is not an easy question to answer. We will take a closer look at it in the next section. But for now, we can show that if $f(x)$ is a polynomial or rational function whose denominator is not 0 at $x = a$ then

$$\lim_{x \to a} f(x) = f(a).$$

When the limit of a function $f(x)$ can be calculated this way at $x = a$, we can say $f$ has the Direct Substitution Property at $x = a$. Polynomials and rational functions have the Direct Substitution Property at every number in their domains. This is worth stating as a theorem. We will prove it too, which is not difficult to do.

**Theorem.** *If $f(x)$ is a polynomial or a rational function and $a$ is in the domain of $f$ then*

$$\lim_{x \to a} f(x) = f(a).$$

*Proof:* First, suppose $f(x)$ is a polynomial

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0.$$

Every real number is in the domain of $f$. So whatever $a$ is, we have

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} (c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0) \\
&= \lim_{x \to a} (c_n x^n) + \lim_{x \to a} (c_{n-1} x^{n-1}) + \cdots + \lim_{x \to a} (c_1 x) + \lim_{x \to a} c_0 && \text{by LL1} \\
&= c_n \lim_{x \to a} x^n + c_{n-1} \lim_{x \to a} x^{n-1} + \cdots + c_1 \lim_{x \to a} x + \lim_{x \to a} c_0 && \text{by LL3} \\
&= c_n a^n + c_{n-1} a^{n-1} + \cdots + c_1 a + c_0 && \text{by LL9 and LL7} \\
&= f(a).
\end{aligned}$$

Now, suppose $f(x) = p(x)/q(x)$ is a rational function, so $p(x)$ and $q(x)$ are polynomials. If $a$ is in the domain of $f$ then $q(a) \neq 0$. So

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} \frac{p(x)}{q(x)} \\
&= \frac{\lim_{x \to a} p(x)}{\lim_{x \to a} q(x)} && \text{by LL5} \\
&= \frac{p(a)}{q(a)}
\end{aligned}$$

since $p(x)$ and $q(x)$ are polynomials and have the Direct Substitution Property at $x = a$. Notice that

$$\lim_{x \to a} q(x) = q(a) \neq 0,$$

which tells us that we were justified in using LL5. □

The direct substitution property of polynomials and rational functions often makes it very easy to calculate their limits. For example,

$$\lim_{x \to -2} (x^3 + 3x^2 - 7) = (-2)^3 + 3(-2)^2 - 7 = -8 + 3(4) - 7 = -3$$

$$\lim_{x \to 5} \frac{3x + 4}{x^2 - 9} = \frac{3(5) + 4}{5^2 - 9} = \frac{19}{16}$$

both by direct substitution. But be careful not to use the direct substitution property for a function that may not have this property. For example, saying that

$$\lim_{x \to 5} \frac{3x + 4}{\sqrt{x^2 - 9}} = \frac{3(5) + 4}{\sqrt{5^2 - 9}} = \frac{19}{\sqrt{16}} = \frac{19}{4}$$

by direct substitution is not justified because the function $f(x) = \frac{3x+4}{\sqrt{x^2-9}}$ is neither a polynomial nor a rational function. Even though $19/4$ is the correct limit, using direct substitution to find it is not justified because we do not know if this function has the direct substitution property. The fact that the number 19/4 we obtained by direct substitution matches the limit we found and justified earlier by using the limit laws suggests that it may well be that $f(x) = \frac{3x+4}{\sqrt{x^2-9}}$ is in fact a function that has the direct substitution property at $x = 5$. In the next section, we will expand our understanding of what kinds of functions do.

Even though we cannot simply substitute $x = 5$ into $f(x) = \frac{3x+4}{\sqrt{x^2-9}}$ to calculate

$$\lim_{x \to 5} \frac{3x+4}{\sqrt{x^2-9}},$$

the direct substitution property does help to calculate this limit quickly and easily. We start, just like before, by using the limit laws:

$$\begin{aligned} \lim_{x \to 5} \frac{3x+4}{\sqrt{x^2-9}} &= \frac{\lim_{x\to 5}(3x+4)}{\lim_{x\to 5}\sqrt{x^2-9}} && \text{by LL5} \\ &= \frac{\lim_{x\to 5}(3x+4)}{\sqrt{\lim_{x\to 5}(x^2-9)}} && \text{by LL11} \end{aligned}$$

Now we can use direct substitution to say

$$\lim_{x \to 5}(3x+4) = 3(5)+4 = 19$$
$$\lim_{x \to 5}(x^2-9) = 5^2 - 9 = 16$$

since $3x + 4$ and $x^2 - 9$ are both polynomials. Hence

$$\lim_{x \to 5} \frac{3x+4}{\sqrt{x^2-9}} = \frac{19}{\sqrt{16}} = \frac{19}{4}.$$

The limit laws and the direct substitution property of polynomials and rational functions are powerful tools to tackle limits, but they are not always enough. For example, we noted above that

$$\lim_{x \to 1} \frac{x^2-1}{x-1}$$

cannot be calculated by direct substitution. Although $\frac{x^2-1}{x-1}$ is a rational function, $x = 1$ is not in the domain because it makes the denominator 0. Trying to use LL5 fails for much the same reason:

$$\lim_{x \to 1} \frac{x^2-1}{x-1} \neq \frac{\lim_{x\to 1}(x^2-1)}{\lim_{x\to 1}(x-1)}$$

because

$$\lim_{x \to 1}(x-1) = 1 - 1 = 0$$

and remember that LL5 is only valid if the limits of both the numerator and the denominator exist and the limit of the denominator is not 0.

We need another idea to tackle such limits. In fact, it is an idea we have already used:

$$\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x + 1 \quad \text{unless } x = 1.$$

When we consider the limit as $x \to 1$, it is exactly what happens when $x = 1$ that we do not care about. This is because $0 < |x - a|$ in the $\delta - \epsilon$ definition. Therefore we can replace $\frac{(x+1)(x-1)}{x-1}$ with $x + 1$ in the limit:

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x+1)(x-1)}{x-1} = \lim_{x \to 1}(x+1) = 1 + 1 = 2$$

by direct substitution since $x + 1$ is a polynomial.

This idea is a really useful tool, spelled out in general in the following theorem. We have already talked about the main idea in the proof.

**Theorem.** *Let $f$ and $g$ be functions of real numbers such that $f(x) = g(x)$ for every $x$ is some neighborhood of $a$ except possibly at $x = a$. Then if either one of*

$$\lim_{x\to a} f(x) \quad \textit{and} \quad \lim_{x\to a} g(x)$$

*exists then so does the other and*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x).$$

*Proof:* Suppose that $f$ and $g$ satisfy the conditions of the theorem and one of the two limits exists. Since the functions $f$ and $g$ play symmetric roles in the statement, it does not matter which limit exists. Let us say

$$\lim_{x\to a} g(x) = L$$

exists. Then we know that for every $\epsilon > 0$ there is a corresponding $\delta_1 > 0$ such that if

$$0 < |x - a| < \delta_1$$

then

$$|g(x) - L| < \epsilon.$$

We also know that $f(x) = g(x)$ for every $x \neq a$ in some neighborhood $(a - \delta_2, a + \delta_2)$ where $\delta_2 > 0$ is some sufficiently small number. Now if $x$ is closer to $a$ then both of the two distances $\delta_1$ and $\delta_2$ then

$$|f(x) - L| = |g(x) - L| < \epsilon.$$

So let $\delta = \min(\delta_1, \delta_2)$. We will show that if

$$0 < |x - a| < \delta$$

then

$$|f(x) - L| < \epsilon.$$

Suppose

$$0 < |x - a| < \delta.$$

Then

$$0 < |x - a| < \delta \leq \delta_1 \implies |g(x) - L| < \epsilon.$$

and

$$0 < |x - a| < \delta \leq \delta_2 \implies f(x) = g(x).$$

Therefore

$$|f(x) - L| = |g(x) - L| < \epsilon.$$

Hence

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} g(x).$$

Now, if it is

$$\lim_{x\to a} f(x) = L$$

that exists, then the same argument with $f$ and $g$ switched shows that

$$\lim_{x\to a} g(x) = L = \lim_{x\to a} f(x).$$

$\square$

Look at Examples 3 and 4 for how this tool can be used. Remember exercise 1.3.2 about a rock that is thrown upward on Mars? We are told that the rock's height is $y = 10t - 1.86t^2$ in meters at time $t$ in seconds. In part (b), of that exercise, you were asked to estimate the instantaneous velocity of the rock at $t = 1$. We can now find that velocity precisely, without estimating. Remember that the average velocity between $t_1$ and $t_2$ is

$$v = \frac{\Delta y}{\Delta t} = \frac{y(t_2) - y(t_1)}{t_2 - t_1},$$

and we said that to calculate the instantaneous velocity at $t = 1$ we would want to let $t$ get closer and closer to 1 but not equal to 1 and see what number the average velocity between 1 and $t$ approaches. What we are saying is that the instantaneous velocity at 1 is

$$\lim_{t\to 1} \frac{y(t) - y(1)}{t - 1} = \lim_{t\to 1} \frac{(10t - 1.86t^2) - \big(10(1) - 1.86(1^2)\big)}{t - 1}$$

This looks complicated, but is in fact not too difficult to calculate.

$$\begin{aligned}
\lim_{t\to 1} \frac{(10t - 1.86t^2) - (10(1) - 1.86(1^2)}{t - 1} &= \lim_{t\to 1} \frac{10t - 1.86t^2 - 10(1) + 1.86(1^2)}{t - 1} \\
&= \lim_{t\to 1} \frac{10(t - 1) - 1.86(t^2 - 1)}{t - 1}
\end{aligned}$$

Remember that $t^2 - 1 = (t + 1)(t - 1)$ and hence

$$\begin{aligned}
\frac{10(t - 1) - 1.86(t^2 - 1)}{t - 1} &= \frac{10(t - 1) - 1.86(t + 1)(t - 1)}{t - 1} \\
&= \frac{(t - 1)\big(10 - 1.86(t + 1)\big)}{t - 1} \\
&= 10 - 1.86(t + 1) \qquad\qquad \text{unless } t = 1.
\end{aligned}$$

But in the limit as $t$ approaches 1, $t$ is never 1, so we can say

$$\begin{aligned}
\lim_{t\to 1} \frac{10(t - 1) - 1.86(t^2 - 1)}{t - 1} &= \lim_{t\to 1} \big(10 - 1.86(t + 1)\big) \\
&= \lim_{t\to 1} (10 - 1.86t - 1.86) \\
&= \lim_{t\to 1} (8.14 - 1.86t) \\
&= 8.14 - 1.86 \\
&= 6.28
\end{aligned}$$

where we used direct substitution since $8.14 - 1.86t$ is a polynomial.

Here is another example of a limit whose calculation takes a little more than just the limit laws:

$$\lim_{x\to 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2}.$$

It is hard to guess what the value of the limit might be or if the limit even exists by thinking about what happens as $x$ approaches 3. For sure, both $\sqrt{x} - \sqrt{2}$ and $x - 2$ get arbitrarily close to 0, but their quotient could be anything: a small number, a large number, or a medium size number. But the same idea we used with $\frac{x^2-1}{x-1}$ can help us here too. Notice that

$$(\sqrt{x} - \sqrt{2})(\sqrt{x} + \sqrt{2}) = \sqrt{x}^2 - \sqrt{2}^2 = x - 2.$$

Hence

$$\frac{\sqrt{x} - \sqrt{2}}{x - 2} = \frac{\sqrt{x} - \sqrt{2}}{(\sqrt{x} - \sqrt{2})(\sqrt{x} + \sqrt{2})}.$$

We want to cancel the factor $\sqrt{x}-\sqrt{2}$, which is fine as long as it is not 0. It would be 0 if

$$\sqrt{x}-\sqrt{2}=0 \iff \sqrt{x}=\sqrt{2} \iff x=2.$$

So

$$\frac{\sqrt{x}-\sqrt{2}}{x-2}=\frac{\sqrt{x}-\sqrt{2}}{(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})}=\frac{1}{\sqrt{x}+\sqrt{2}} \quad \text{unless } x=2.$$

But $x \neq 2$ as $x \to 2$. By the theorem we proved above,

$$\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2}=\lim_{x\to 2}\frac{1}{\sqrt{x}+\sqrt{2}}$$

Now we can deploy the limit laws:

$$\begin{aligned}
\lim_{x\to 2}\frac{1}{\sqrt{x}+\sqrt{2}} &= \frac{\lim_{x\to 2} 1}{\lim_{x\to 2}(\sqrt{x}+\sqrt{2})} && \text{by LL5}\\
&= \frac{1}{\lim_{x\to 2}\sqrt{x}+\lim_{x\to 2}\sqrt{2}} && \text{by LL7 and LL1}\\
&= \frac{1}{\sqrt{2}+\sqrt{2}} && \text{by LL10 and LL7}\\
&= \frac{1}{2\sqrt{2}}
\end{aligned}$$

There is another way to do this, which is also worth learning. For any real number $x$,

$$\sqrt{x}\geq 0 \implies \sqrt{x}+\sqrt{2}\geq\sqrt{2}.$$

In particular, $\sqrt{x}+\sqrt{2}\neq 0$, and so

$$\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}=1.$$

Therefore

$$\begin{aligned}
\frac{\sqrt{x}-\sqrt{2}}{x-2} &= \frac{\sqrt{x}-\sqrt{2}}{x-2}\,\frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}\\
&= \frac{(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})}{(x-2)(\sqrt{x}+\sqrt{2})}\\
&= \frac{\sqrt{x}^2-\sqrt{2}^2}{(x-2)(\sqrt{x}+\sqrt{2})}\\
&= \frac{x-2}{(x-2)(\sqrt{x}+\sqrt{2})}\\
&= \frac{1}{\sqrt{x}+\sqrt{2}} && \text{unless } x=2
\end{aligned}$$

Now, just like before,

$$\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2}=\lim_{x\to 2}\frac{1}{\sqrt{x}+\sqrt{2}},$$

and the limit on the right-hand side can be found using the limit laws. You may remember seeing this idea when you learned how to rationalize the denominator of a fraction. In this case, it does not matter if the square root is in the numerator or in the denominator. Remember that $\sqrt{x}+\sqrt{2}$ is called the conjugate of $\sqrt{x}-\sqrt{2}$. Multiplying the numerator and the denominator by this conjugate helped us simplify the fraction. This same idea is used in Example 5 in the textbook.

The limit laws have certainly made our lives easier and will continue to do so. But there is a price to pay: we need to prove them. We will not prove all of them, but we will prove a few to develop an understanding of why they hold. The easiest one to prove is LL7, which was also exercise 1.3.38 on your homework:

$$\lim_{x \to a} c = c$$

where $a$ and $c$ are any real numbers.

*Proof:* We need to show that for every $\epsilon > 0$ there is a corresponding $\delta > 0$ such that if

$$0 < |x - a| < \delta$$

then

$$|c - c| < \epsilon.$$

Since $c - c = 0$ and $|0| = 0 < \epsilon$, it does not actually matter what $x$ is. So any choice of $\delta > 0$ is a perfectly good one. E.g. if $\delta = 1$, it is certainly true that if

$$0 < |x - a| < \delta = 1$$

then

$$|c - c| = |0| = 0 < \epsilon.$$

$\square$

It is not much harder to prove LL8:

$$\lim_{x \to a} x = a$$

where $a$ is any real number.

*Proof:* We need to show that for every $\epsilon > 0$ there is a corresponding $\delta > 0$ such that if

$$0 < |x - a| < \delta$$

then

$$|x - a| < \epsilon.$$

The obvious choice is $\delta = \epsilon$. Now if

$$0 < |x - a| < \delta = \epsilon$$

then

$$|x - a| < \epsilon.$$

$\square$

The next one we will do is LL3: if $\lim_{x \to a} f(x)$ exists then

$$\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$$

for any real number $c$.

*Proof:* Let

$$L = \lim_{x \to a} f(x).$$

We will show

$$\lim_{x \to a} [cf(x)] = cL.$$

First, if $c = 0$ then

$$\lim_{x \to a} [cf(x)] = \lim_{x \to a} 0 = 0 = 0(L)$$

by LL7. Now, suppose $c \neq 0$. We need to show that for every $\epsilon > 0$ there is a corresponding $\delta > 0$ such that if

$$0 < |x - a| < \delta$$

then

$$|cf(x) - cL| < \epsilon.$$

So let $\epsilon > 0$. We want

$$|cf(x) - cL| < \epsilon.$$

Note

$$|cf(x) - cL| = |c\big(f(x) - L\big)| = |c||f(x) - L|.$$

So we really want

$$|c||f(x) - L| < \epsilon \implies |f(x) - L| < \frac{\epsilon}{|c|},$$

where we could divide both sides of the inequality by $|c|$ because we know $c \neq 0$ and hence $|c|$ is positive. Since

$$\lim_{x \to a} f(x) = L,$$

there must exist some $\delta > 0$ such that if

$$0 < |x - a| < \delta$$

then

$$|f(x) - L| < \frac{\epsilon}{|c|}.$$

We will show that this $\delta$ is what we are looking for, that is if

$$0 < |x - a| < \delta$$

then

$$|cf(x) - cL| < \epsilon.$$

Suppose

$$0 < |x - a| < \delta.$$

Then

$$|cf(x) - cL| = |c||f(x) - L| < |c|\,\frac{\epsilon}{|c|} = \epsilon,$$

which is what we wanted to show. □

Next on my list is LL1, but we first need a result, which is very useful when constructing $\delta - \epsilon$ arguments:

**Theorem.** *(The Triangle Inequality) For any $a, b, \in \mathbb{R}$,*

$$|a + b| \leq |a| + |b|.$$

One way to show this is true is to split it into four cases depending on whether $a \geq 0$ or $a < 0$ and whether $b \geq 0$ or $b < 0$. In fact, I suggest you try these four cases for yourself with specific numbers for $a$ and $b$. I will not spell out the whole proof here because it is spelled out quite clearly in Appendix C in your textbook.

We are now ready to prove LL1:

$$\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

if the limits of $f$ and $g$ on the right-hand side both exist.

*Proof:* Let

$$L_1 = \lim_{x \to a} f(x) \quad \text{and} \quad L_2 = \lim_{x \to a} g(x).$$

We want to show

$$\lim_{x \to a}[f(x) + g(x)] = L_1 + L_2.$$

We need to show that for every $\epsilon > 0$ there is a corresponding $\delta > 0$ such that if

$$0 < |x - a| < \delta$$

then

$$\big|\big(f(x)+g(x)\big)-(L_1+L_2)\big|<\epsilon.$$

So let $\epsilon>0$. We want

$$\big|\big(f(x)+g(x)\big)-(L_1+L_2)\big|<\epsilon.$$

By the Triangle Inequality,

$$\big|\big(f(x)+g(x)\big)-(L_1+L_2)\big|=|f(x)+g(x)-L_1-L_2|=|f(x)-L_1+g(x)-L_2|\le|f(x)-L_1|+|g(x)-L_2|.$$

Notice that if we can make both $|f(x)-L_1|+$ and $|g(x)-L_2|$ smaller than $\epsilon/2$ then their sum will be smaller than $\epsilon$. Since

$$\lim_{x\to a} f(x)=L_1,$$

there must exist some $\delta_1>0$ such that if

$$0<|x-a|<\delta_1$$

then

$$|f(x)-L_1|<\frac{\epsilon}{2}.$$

Similarly, since

$$\lim_{x\to a} g(x)=L_2,$$

there must exist some $\delta_2>0$ such that if

$$0<|x-a|<\delta_2$$

then

$$|g(x)-L_2|<\frac{\epsilon}{2}.$$

The reason we used $\delta_1$ and $\delta_2$ is because there is no reason to believe that they have the same value. While we know that we can make $|f(x)-L_1|<\epsilon/2$ if $x$ is sufficiently close to $a$ and we can also make $|g(x)-L_2|<\epsilon/2$ is $x$ is sufficiently close to $a$, what is sufficiently close likely means different things for $f$ and $g$. Now, we want to make $\delta$ equal to the smaller of $\delta_1$ and $\delta_2$. That is set

$$\delta=\min(\delta_1,\delta_2).$$

We will show that this $\delta$ is sufficiently small so that if

$$0<|x-a|<\delta$$

then

$$\big|\big(f(x)+g(x)\big)-(L_1+L_2)\big|<\epsilon.$$

Suppose

$$0<|x-a|<\delta.$$

Then

$$\begin{aligned}\big|\big(f(x)+g(x)\big)-(L_1+L_2)\big|&=|f(x)+g(x)-L_1-L_2|\\&=|f(x)-L_1+g(x)-L_2|\le|f(x)-L_1|+|g(x)-L_2|\\&<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon,\end{aligned}$$

which is what we wanted to show. $\square$

This last argument is a little complex, but we learned a new trick: the divide and conquer strategy by making $f(x)$ and $g(x)$ get closer to $L_1$ and $L_2$ respectively than half of $\epsilon$.

Now, to prove LL2 that

$$\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

if the limits of $f$ and $g$ on the right-hand side both exist, we could do almost the same thing. In fact all we would have to do is switch a few $+$ signs to $-$ signs and vice versa in the above argument. Or we can do it even more easily by combining LL1 and LL3.

*Proof:*

$$\begin{aligned}
\lim_{x\to a}[f(x) - g(x)] &= \lim_{x\to a}[f(x) + (-1)g(x)] \\
&= \lim_{x\to a} f(x) + \lim_{x\to a}[(-1)g(x)] && \text{by LL1} \\
&= \lim_{x\to a} f(x) + (-1)\lim_{x\to a} g(x) && \text{by LL3} \\
&= \lim_{x\to a} f(x) - \lim_{x\to a} g(x)
\end{aligned}$$

$\square$

To prove LL4 and LL5, we could use similar ideas to the ones we used for proving LL1. The arguments are a little more complicated than the one for LL1. We will not cover them in class, but if curiosity does not let you rest, they are given in Appendix C in your textbook.

On to LL6:

$$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$$

for any $n \in \mathbb{Z}^+$ if the limit of $f$ on the right-hand side exists.

*Proof:*

$$\begin{aligned}
\lim_{x\to a}[f(x)]^n &= \lim_{x\to a}[\underbrace{f(x)f(x)\cdots f(x)}_{n \text{ factors}}] \\
&= \underbrace{\left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} f(x)\right]\cdots\left[\lim_{x\to a} f(x)\right]}_{n \text{ factors}} && \text{by LL4} \\
&= \left[\lim_{x\to a} f(x)\right]^n
\end{aligned}$$

$\square$

LL9 is an easy consequence of LL6 and LL7:

$$\lim_{x\to a} x^n = a^n$$

for any $n \in \mathbb{Z}^+$.

*Proof:* Let $f(x) = x$. Then

$$\begin{aligned}
\lim_{x\to a} x^n &= \lim_{x\to a}[f(x)]^n \\
&= \left[\lim_{x\to a} f(x)\right]^n && \text{by LL6} \\
&= \left[\lim_{x\to a} x\right]^n \\
&= a^n && \text{by LL7}
\end{aligned}$$

$\square$

That is good enough. We will not prove LL10 and LL11, at least for now. Here is another useful result we know already, but we have not proved it rigorously:

**Theorem.** *Let $f$ be a function of real numbers and let $a \in \mathbb{R}$. Then*

$$\lim_{x \to a} f(x) = L$$

*if and only if*

$$\lim_{x \to a^+} f(x) = L \quad \textit{and} \quad \lim_{x \to a^-} f(x) = L.$$

*Proof:* First, suppose

$$\lim_{x \to a} f(x) = L.$$

We want to show that

$$\lim_{x \to a^+} f(x) = L \quad \text{and} \quad \lim_{x \to a^-} f(x) = L.$$

We will tackle the right limit first. We need to prove that for every $\epsilon > 0$ there is a corresponding $\delta > 0$ such that if

$$a < x < a + \delta$$

then

$$|f(x) - L| < \epsilon.$$

So let $\epsilon > 0$. Since

$$\lim_{x \to a} f(x) = L,$$

we know there is a $\delta > 0$ such that if

$$|x - a| < \delta$$

then

$$|f(x) - L| < \epsilon.$$

We can use the same $\delta$ for the right limit. If

$$a < x < a + \delta$$

then it is certainly true that $x$ is closer to $a$ then a distance of $\delta$, that is

$$|x - a| < \delta$$

and hence

$$|f(x) - L| < \epsilon.$$

This proves

$$\lim_{x \to a^+} f(x) = L.$$

An analogous argument with the same value of $\delta$ shows

$$\lim_{x \to a^-} f(x) = L.$$

Now, suppose

$$\lim_{x \to a^+} f(x) = L \quad \text{and} \quad \lim_{x \to a^-} f(x) = L.$$

We want to prove

$$\lim_{x \to a} f(x) = L.$$

Since

$$\lim_{x \to a^+} f(x) = L,$$

there must exist some $\delta_1 > 0$ such that if

$$a < x < a + \delta_1$$

then

$$|f(x) - L| < \epsilon.$$

Similarly, since

$$\lim_{x \to a^-} f(x) = L,$$

there must exist some $\delta_2 > 0$ such that if

$$a - \delta_2 < x < a$$

then

$$|f(x) - L| < \epsilon.$$

Notice that we once again used $\delta_1$ and $\delta_2$, just like in the proof of LL1 earlier, because there is no reason to believe that they have the same value. We know that we can make $|f(x) - L| < \epsilon$ if $x$ is sufficiently close to $a$ on the right and we can also make $|f(x) - L| < \epsilon$ if $x$ is sufficiently close to $a$ on the left, but how close is sufficiently close likely means different things on the left and the right. Now, we make $\delta$ equal to the smaller of $\delta_1$ and $\delta_2$:

$$\delta = \min(\delta_1, \delta_2).$$

We will show that this $\delta$ is sufficiently small that if

$$0 < |x - a| < \delta$$

then

$$|f(x) - L| < \epsilon.$$

Suppose

$$0 < |x - a| < \delta.$$

Then either

$$a < x < a + \delta$$

or

$$a - \delta < x < a.$$

In the first case, it is also true that

$$a < x < a + \delta \implies a < x < a + \delta_1 \implies |f(x) - L| < \epsilon.$$

In the other case, it is also true that

$$a - \delta < x < a \implies a - \delta_2 < x < a \implies |f(x) - L| < \epsilon.$$

In either case, we can conclude

$$|f(x) - L| < \epsilon,$$

which is what we wanted to show. □

This result is typically used when we want to find the limit of a piecewise-defined function at the boundary of the pieces, or a function with an absolute value in it. It can be used to find the limit if the limit exists, like in Example 6 in the textbook, or to show that the limit does not exist, like in Examples 7 and 8. In fact, we have already look at the limit in Example 7 in class and in the notes for Section 1.3.

There are two more results in this section to complete our set of tools. Both are quite easy to prove, but they do require $\delta - \epsilon$ arguments.

**Theorem.** *Let $f$ and $g$ be functions of real numbers such that*

$$\lim_{x\to a} f(x) \quad \textit{and} \quad \lim_{x\to a} g(x)$$

*both exist. If there is a neighborhood of $a$ such that $f(x) \le g(x)$ for all $x$ in that neighborhood except possibly at $x = a$, then*

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

*Proof:* Suppose that the conclusion that

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

is false. We will show that this leads to a contradiction. Let

$$L_1 = \lim_{x\to a} f(x) \quad \text{and} \quad L_2 = \lim_{x\to a} g(x).$$

So we are assuming that $L_1 \le L_2$ is false. Then $L_1 > L_2$ must be true. Therefore $L_1 - L_2$ is a positive number. Let $\epsilon = (L_1 - L_2)/2$, which is then also positive. Therefore there exists $\delta_1 > 0$ such that if

$$0 < |x - a| < \delta_1$$

then

$$|f(x) - L_1| < \epsilon \implies L_1 - \epsilon < f(x) < L_1 + \epsilon.$$

In particular,

$$L_1 - \epsilon < f(x) \implies L_1 - \frac{L_1 - L_2}{2} < f(x) \implies L_1 - \frac{L_1}{2} + \frac{L_2}{2} < f(x) \implies \frac{L_1}{2} + \frac{L_2}{2} < f(x).$$

Similarly, there exists $\delta_2 > 0$ such that if

$$0 < |x - a| < \delta_2$$

then

$$|g(x) - L_2| < \epsilon \implies L_2 - \epsilon < g(x) < L_2 + \epsilon.$$

In particular,

$$g(x) < L_2 + \epsilon \implies g(x) < L_2 + \frac{L_1 - L_2}{2} \implies g(x) < L_2 + \frac{L_1}{2} - \frac{L_2}{2} \implies g(x) < \frac{L_1}{2} + \frac{L_2}{2}.$$

Finally, we know that there is some neighborhood of $a$, namely some interval $(a - \delta_3, a + \delta_3)$ for some $\delta_3 > 0$ such that if $x \in (a - \delta_3, a + \delta_3)$ then $f(x) \le g(x)$. Another way to say that is if

$$0 < |x - a| < \delta_3$$

then

$$f(x) \le g(x).$$

Now, all this means that if $x \ne a$ is closer to $a$ then $\delta_1$, $\delta_2$, and $\delta_3$ that is if

$$0 < |x - a| < \min(\delta_1, \delta_2, \delta_3)$$

then

$$g(x) < \frac{L_1}{2} + \frac{L_2}{2} < f(x)$$

and

$$f(x) \le g(x).$$

But we clearly cannot have $g(x) < f(x)$ and $f(x) \le g(x)$ at the same time. $\square$

Finally, the last result is so useful that it has its own name. Actually, it has several names: Squeeze Theorem, Sandwich Theorem, Pinching Theorem, and sometimes Police Principle.

**Theorem.** *Let $f$, $g$, and $h$ be functions of real numbers such that*

$$f(x) \leq g(x) \leq h(x)$$

*for every $x$ in some neighborhood of $a$ except possibly at $x = a$. If*

$$\lim_{x \to a} f(x) \quad \textit{and} \quad \lim_{x \to a} h(x)$$

*both exist and*

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

*then*

$$\lim_{x \to a} g(x) = L.$$

What the theorem says should make good intuitive sense: if the value of $f(x)$ and the value of $h(x)$ can both be made to get close to $L$ when $x$ is close to $a$, then the value of $g(x)$, which is sandwiched between $f(x)$ and $h(x)$ must also get close to $L$. Figures 6 and 7 nicely capture this intuitive idea in visual form. Take a look at Example 9 for how the theorem can be used before we show why the theorem holds. I will do a similar example further down. Note that this theorem is different from the previous one in that we do not need to know that the limit of $g$ at $x = a$ exists to be able to conlude

$$L = \lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \leq \lim_{x \to a} h(x) = L.$$

*Proof:* We need to show that for every $\epsilon > 0$ there is a corresponding $\delta > 0$ such that if

$$0 < |x - a| < \delta$$

then

$$|g(x) - L| < \epsilon.$$

First, since

$$\lim_{x \to a} f(x) = L$$

there is some $\delta_1 > 0$ such that if

$$0 < |x - a| < \delta_1$$

then

$$|f(x) - L| < \epsilon.$$

Similarly, since

$$\lim_{x \to a} h(x) = L$$

there is some $\delta_2 > 0$ such that if

$$0 < |x - a| < \delta_2$$

then

$$|h(x) - L| < \epsilon.$$

Finally, there is some $\delta_3$ such that if

$$0 < |x - a| < \delta_3$$

then

$$f(x) \leq g(x) \leq h(x).$$

Let $\delta$ be the smallest of $\delta_1$, $\delta_2$, and $\delta_3$ that is

$$\delta = \min(\delta_1, \delta_2, \delta_3).$$

We will show that if

$$0 < |x - a| < \delta$$

then

$$|g(x) - L| < \epsilon.$$

Suppose

$$0 < |x - a| < \delta.$$

Then

$$0 < |x - a| < \delta \leq \delta_1 \implies |f(x) - L| < \epsilon \implies L - \epsilon < f(x) < L + \epsilon.$$

Also

$$0 < |x - a| < \delta \leq \delta_2 \implies |h(x) - L| < \epsilon \implies L - \epsilon < h(x) < L + \epsilon.$$

Finally,

$$0 < |x - a| < \delta \leq \delta_3 \implies f(x) \leq g(x) \leq h(x).$$

Putting the three pieces together, we get

$$L - \epsilon < f(x) \leq g(x) \leq h(x) < L + \epsilon,$$

and so

$$|g(x) - L| < \epsilon.$$

□

The Limit Laws and the other results we learned in this section also work for one-sided limits. This is because one-sided limits are just special cases of the general limit. Here is ane example in which we will use them for one-sided limits to show that

$$\lim_{x \to 0} \left[ x^3 \cos\left(\frac{\pi}{x}\right) \right] = 0.$$

First note that we cannot just use LL4 to do this:

$$\lim_{x \to 0} \left[ x^3 \cos\left(\frac{\pi}{x}\right) \right] \neq \left[ \lim_{x \to 0} x^3 \right] \left[ \lim_{x \to 0} \cos\left(\frac{\pi}{x}\right) \right]$$

since

$$\lim_{x \to 0} \cos\left(\frac{\pi}{x}\right)$$

does not exist because $\cos\left(\frac{\pi}{x}\right)$ oscillates back and forth between -1 and 1 no matter how close $x$ is to 0. In fact, since $-1 \leq \cos\left(\frac{\pi}{x}\right) \leq 1$, the value of $x^3 \cos\left(\frac{\pi}{x}\right)$ is always sandwiched between $-x^3$ and $x^3$. But we need to be careful about how because it depends on the sign of $x$ which of these is the upper bound and which is the lower bound. If $x > 0$ then $x^3$ is also positive, so

$$-1 < \cos\left(\frac{\pi}{x}\right) < 1 \implies -x^3 < x^3 \cos\left(\frac{\pi}{x}\right) < x^3.$$

Now,

$$\lim_{x \to 0^+} x^3 = 0^3 = 0$$

by LL9 or by the direct subtitution property of polynomials. Also,

$$\lim_{x \to 0^+} (-x^3) = -0^3 = 0.$$

Therefore

$$\lim_{x \to 0^+} \left[ x^3 \cos\left(\frac{\pi}{x}\right) \right] = 0.$$

by the Squeeze Theorem. Similarly, if $x < 0$ then $x^3$ is negative, and so

$$-1 < \cos\left(\frac{\pi}{x}\right) < 1 \implies -x^3 > x^3 \cos\left(\frac{\pi}{x}\right) > x^3.$$

Now,

$$\lim_{x \to 0^-} x^3 = 0^3 = 0$$

by LL9 or by the direct subtitution property of polynomials. Also,

$$\lim_{x \to 0^-} (-x^3) = -0^3 = 0.$$

Therefore

$$\lim_{x \to 0^-} \left[ x^3 \cos\left(\frac{\pi}{x}\right) \right] = 0.$$

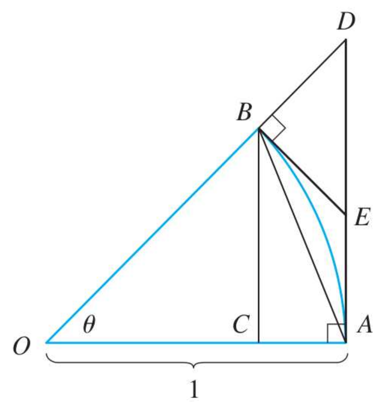
by the Squeeze Theorem. Finally, since the left and the right limits are both 0,

$$\lim_{x \to 0} \left[ x^3 \cos\left(\frac{\pi}{x}\right) \right] = 0.$$

We will close this section by looking at some limits of trigonometric functions. If you think about the unit circle, it should make good sense that

$$\lim_{x \to 0} \sin(x) = 0 \quad \text{and} \quad \lim_{x \to 0} \cos(x) = 1.$$

While this is hardly a rigorous argument, it gives valuable intuition about why these limits are 0 and 1. We can make the thinking more rigorous by doing some geometry. Here is Figure 8(a) from your textbook:




The light blue wedge in this diagram is a part of the unit circle. Point $B$ is the point that corresponds to the angle $\theta$ on the unit circle. The vertical coordinate of $B$ is therefore $\sin(\theta)$. This is the same as the length of the line segment $BC$. Since $BC$ is also one of the legs in the right triangle $ABC$, it must be shorter than the hypotenuse $AB$. But $AB$ is also a secant in the unit circle and is therefore shorter than the corresponding arc $AB$. In fact, the shortest path between points $A$ and $B$ is the line segment $AB$. The length of arc $AB$ is given by the usual arc length formula $r\theta$ where $r = 1$ in this case. So the length of arc $AB$ is in fact $\theta$. Putting these pieces together tells us

$$\sin(\theta) = BC < AB < \text{arc } AB = \theta.$$

Since $BC$ is the side of a triangle, it is also clear that $0 < BC$. This of course only works if $\theta$ is a small (smaller than $\pi/2$) positive angle, exactly as it looks in the diagram. But that is enough for our purposes. So if $\theta$ is a small positive angle, then

$$0 < \sin(\theta) < \theta,$$

that is $\sin(\theta)$ is sandwiched between 0 and $\theta$. Now,

$$\lim_{\theta \to 0^+} 0 = 0$$

by LL7, and

$$\lim_{\theta \to 0^+} \theta = 0$$

by LL8. By the Squeeze Theorem,

$$\lim_{\theta \to 0^+} \sin(\theta) = 0.$$

What if $\theta < 0$? Now, $-\theta$ is positive, so by the argument above,

$$0 < \sin(-\theta) < -\theta \implies 0 < -\sin(\theta) < -\theta \implies 0 > \sin(\theta) > \theta.$$

We can still use the Squeeze Theorem by noting that

$$\lim_{\theta \to 0^-} 0 = 0$$

by LL7, and

$$\lim_{\theta \to 0^-} \theta = 0$$

by LL8, and therefore

$$\lim_{\theta \to 0^-} \sin(\theta) = 0.$$

We can now conclude that

$$\lim_{x \to 0} \sin(x) = 0.$$

That

$$\lim_{x \to 0} \cos(x) = 1$$

can be shown similarly, using this time that

$$0 < AC < AB < \text{arc } AB = \theta,$$

and

$$\cos(\theta) = OC = OA - AC = 1 - AC$$

and hence

$$1 - 0 > 1 - AC > 1 - \theta \implies 1 > \cos(\theta) > 1 - \theta.$$

You can now use the Squeeze Theorem to show that

$$\lim_{\theta \to 0^+} \cos(\theta) = 1,$$

and the even symmetry of the cosine function to show that the left limit is also 1, and hence the two-sided limit is 1. Try writing down this argument from beginning to end on your own.

Interestingly, it is now quite easy to show that

$$\lim_{x \to a} \sin(x) = \sin(a)$$

for any $a \in \mathbb{R}$. That is the sine function also has the direct substitution property at any real number. The idea is to introduce another variable $h$ such that $h = x - a$. If $x$ is close to $a$ then $h$ is close to 0, and vice versa, if $h$ is close to 0 then $x$ is close to $a$. In fact,

$$|h - 0| = |h| = |x - a|,$$

that is $h$ is exactly the same distance from 0 as $x$ is from $a$. Since $x = a+h$ and $\sin(x) = \sin(a+h)$, it follows that

$$\lim_{x \to a} \sin(x) = \lim_{h \to 0} \sin(a + h).$$

We now need the addition formula for the sine function (remember from trigonometry or pre-calculus?):

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

for any $\alpha, \beta \in \mathbb{R}$. So

$$\sin(a + h) = \sin(a)\cos(h) + \cos(a)\sin(h).$$

Therefore

$$\begin{aligned}
\lim_{h \to 0} \sin(a + h) &= \lim_{h \to 0} [\sin(a)\cos(h) + \cos(a)\sin(h)] \\
&= \lim_{h \to 0} [\sin(a)\cos(h)] + \lim_{h \to 0} [\cos(a)\sin(h)] && \text{by LL1}
\end{aligned}$$

Notice that $\sin(a)$ and $\cos(a)$ do not depend on $h$, so they are constant factors. Hence

$$\begin{aligned}\lim_{h\to 0}[\sin(a)\cos(h)] + \lim_{h\to 0}[\cos(a)\sin(h)] &= \sin(a)\lim_{h\to 0}\cos(h) + \cos(a)\lim_{h\to 0}\sin(h) \qquad \text{by LL3}\\ &= \sin(a)\cdot 1 + \cos(a)\cdot 0\\ &= \sin(a)\end{aligned}$$

by what we already know about the limits of $\sin(x)$ and $\cos(x)$ as $x \to 0$.

One can show similarly that cosine also has the direct substitution property at any real number. I will leave it to you to come up with the argument. You will need the addition formula for the cosine function.

In the previous section, we looked at the limit of the function $f(x) = \frac{\sin(x)}{x}$ as $x \to 0$. Based on a table of values and the graph, we guessed that

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$$

but we did not have the tools show that our guess is in fact correct. We will now take a more rigorous look at this limit.

We have already argued above that if $x$ is a positive number close to 0 then $\sin(x) < x$. Hence

$$\frac{\sin(x)}{x} < 1.$$

We will now find a lower bound. Turn your attention to Figure 8(a) from the textbook again. Your geometric intuition likely tells you that walking from point $A$ to point $B$ via the line segment $AE$ followed by the line segment $EB$ is a longer path than following the arc $AB$. If that is not clear, it may help to look at Figure 8(b) in your textbook to see how a polygon circumscribed around a circle, which consists of such broken line segment pieces as the path along $AEB$, has a longer perimeter than the circumference of the circle. Based on such geometric intuition, we can say

$$\text{arc } AB < AE + EB$$

Now notice that line segment $EB$ is shorter than line segment $ED$ because $ED$ is the hypotenuse of the right triangle $EBD$ whereas $EB$ is only one of the legs in that triangle. Hence

$$\text{arc } AB < AE + EB < AE + ED = AD.$$

Remember that when we looked at trigonometric functions earlier this semester, we saw that $AD$ is exactly the way to visualize $\tan(\theta)$ because

$$\tan(\theta) = \frac{AD}{OA} = \frac{AD}{1} = AD.$$

Putting these pieces together, we get

$$\theta = \text{arc } AB < AD = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.$$

Since $\theta$ is a small positive angle, $\cos(\theta)$ is also positive. Hence

$$\theta < \frac{\sin(\theta)}{\cos(\theta)} \implies \theta\cos(\theta) < \sin(\theta) \implies \cos(\theta) < \frac{\sin(\theta)}{\theta}.$$

Hence if $x$ is positive and close to 0,

$$\cos(x) < \frac{\sin(x)}{x} < 1.$$

We already know

$$\lim_{x\to 0^+}\cos(x) = 1.$$

We also know

$$\lim_{x\to 0^+} 1 = 1$$

by LL7. We can now conclude

$$\lim_{x\to 0^+} \frac{\sin(x)}{x} = 1$$

by the Squeeze Theorem. In fact, it does not matter if $x$ is positive or negative because

$$\frac{\sin(-x)}{-x} = \frac{-\sin(x)}{-x} = \frac{\sin(x)}{x}$$

for all real numbers $x$. In other words, $f(x) = \frac{\sin(x)}{x}$ is an even function. Therefore

$$\lim_{x\to 0^-} \frac{\sin(x)}{x} = \lim_{x\to 0^+} \frac{\sin(x)}{x} = 1.$$

It now follows that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

Equipped with what we have learned about the limits of trigonometric functions, we can now work out all kinds of limits that involve them. Look at Examples 10 and 11 in the textbook. I believe the argument in Example 10 actually has a typo in it. It probably means to go like this:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(7x)}{4x} &= \lim_{x\to 0} \left[\frac{7}{4}\frac{\sin(7x)}{7x}\right] \\ &= \frac{7}{4}\lim_{x\to 0}\frac{\sin(7x)}{7x} && \text{by LL3}\end{aligned}$$

Now, $x \to 0$ exactly if $7x \to 0$. So we can substitute $\theta = 7x$ into the limit:

$$\lim_{x\to 0} \frac{\sin(7x)}{7x} = \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1.$$

Hence

$$\lim_{x\to 0} \frac{\sin(7x)}{4x} = \frac{7}{4}\lim_{x\to 0} \frac{\sin(7x)}{7x} = \frac{7}{4}.$$

Another such example would be to find

$$\lim_{x\to 0} \frac{\sin(7x)}{\sin(4x)}.$$

It may be tempting to use LL5, but that does not help because the limit of the denominator turns out to be 0. Here is why. If $x \to 0$ then $4x \to 0$ as well. So we can substitute $h = 4x$ as in

$$\lim_{x\to 0} \sin(4x) = \lim_{h\to 0} \sin(h) = 0.$$

What we can do instead is multiply the numerator and the denominator by $4x$, which we know is not 0 because $x \to 0$ means $x$ is getting close to 0 but $x$ is not 0. So

$$\frac{\sin(7x)}{\sin(4x)} = \frac{\sin(7x)}{\sin(4x)}\frac{4x}{4x} = \frac{\sin(7x)}{4x}\frac{4x}{\sin(4x)} = \frac{\frac{\sin(7x)}{4x}}{\frac{\sin(4x)}{4x}}.$$

Now

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(7x)}{\sin(4x)} &= \lim_{x\to 0} \frac{\frac{\sin(7x)}{4x}}{\frac{\sin(4x)}{4x}} \\ &= \frac{\lim_{x\to 0}\frac{\sin(7x)}{4x}}{\lim_{x\to 0}\frac{\sin(4x)}{4x}} && \text{by LL5}\end{aligned}$$

We already know
$$\lim_{x\to 0} \frac{\sin(7x)}{4x} = \frac{7}{4}.$$
For the second factor, $x \to 0$ if and only if $4x \to 0$, so we can substitute $\theta = 4x$ in
$$\lim_{x\to 0} \frac{\sin(4x)}{4x} = \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1.$$
Therefore
$$\lim_{x\to 0} \frac{\sin(7x)}{\sin(4x)} = \frac{7}{4}.$$