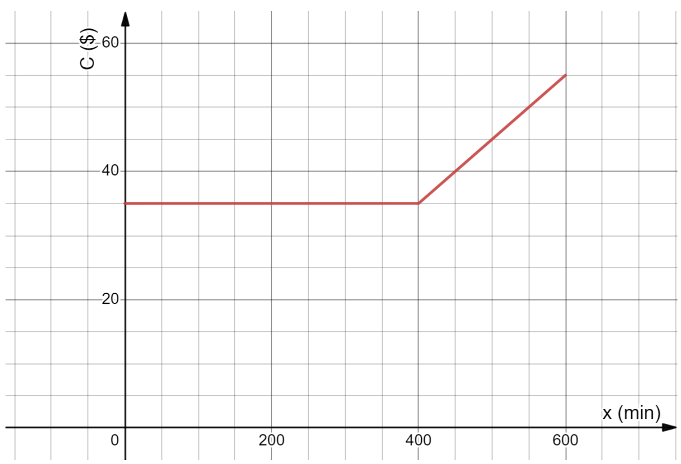
MCS 118 Exam 1 Solutions

1. (10 pts) A cell phone plan has a basic charge of \$35 a month. The plan includes 400 free minutes and charges 10 cents for each additional minute of usage. Write the monthly cost C as a function of the number x of minutes used and graph C as a function of x for $0 \le x \le 600$. Make sure your graph is labeled with a proper scale and units and of course, you must explain/justify your work.

If $0 \le x \le 400$, then $C(x) = 35$. If $x > 400$, then we have used $x - 400$ minutes in addition to the ones included in the plan, and so we would have to pay another $0.10(x - 400)$ dollars for those, plus of course the \$35 basic charge. So

$$C(x) = \begin{cases} 35 & \text{if } 0 \le x \le 400 \\ 35 + 0.10(x - 400) & \text{if } 400 < x \end{cases}$$

Here is the graph of this piecewise defined function:



2. (5 pts each) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions.
   (a) If f and g are both odd functions, is the product fg odd? If you think it is, find an argument to prove that it always is; if you do not think it is, find a counterexample.

   We know $f(-x) = -f(x)$ and $g(-x) = -g(x)$ for all $x \in \mathbb{R}$. So
   $$(fg)(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = (fg)(x)$$
   for all $x \in \mathbb{R}$. This shows fg is an even function. Now, that does not immediately mean that fg cannot be an odd function–remember that it is possible for a function to be both even and odd. But we can easily show by an example that fg is not in general odd. Let $f(x) = g(x) = x$. Then f and g are both odd since $f(-x) = -x = -f(x)$ for all $x \in \mathbb{R}$. Now, $(fg)(x) = x^2$, and that is not an odd function because $(fg)(-x) = (-x)^2 = x^2 \neq -x^2$ in general. For example, $(fg)(-2) = (-2)^2 = 4 \neq -2^2$.

   (b) What if f is even and g is odd? Is fg odd then? Or is it even? Or is it neither? Do not forget to justify your answer.

   We know $f(-x) = f(x)$ and $g(-x) = -g(x)$ for all $x \in \mathbb{R}$. So
   $$(fg)(-x) = f(-x)g(-x) = f(x)\big(-g(x)\big) = -f(x)g(x) = -(fg)(x)$$
   for all $x \in \mathbb{R}$. This shows fg is an odd function.

3. (5 pts each) Let

$$f(x) = \frac{2x+5}{x-3}.$$

(a) Find the largest possible subset of the real numbers that could be the domain of f.

Note that $f(x)$ has no value when the denominator is 0, which happens when

$$x - 3 = 0 \iff x = 3.$$

For any real number $x \neq 3$, both the numerator and the denominator evaluate to some real number and the denominator is not 0. Hence the fraction has a legitimate value. Therefore

$$D(f) = (-\infty, 3) \cup (3, \infty).$$

(b) Using the largest possible domain in $\mathbb{R}$, what would be the range of f?

Hint: a number y will be in the domain if there is an x that satisfies the equation $y = \frac{2x+5}{x-3}$.

The range of f consists of those real numbers y the result from evaluating f at some input $x \in (infty, 3) \cup (3, \infty)$. That is y is in the range of f if there exists a number $x \in (infty, 3) \cup (3, \infty)$ such that

$$y = f(x) = \frac{2x+5}{x-3}.$$

We can try solve this equation for x:

$$\begin{aligned} y &= \frac{2x+5}{x-3} && \text{multiply by } x-3\text{, which we know is not } 0 \\ y(x-3) &= 2x+5 \\ yx - 3y &= 2x+5 && \text{add } 3y \text{ and subtract } 2x \\ yx - 2x &= 3y+5 \\ (y-2)x &= 3y+5 && \text{divide by } y-2 \text{ if it is not } 0 \\ x &= \frac{3y+5}{y-2} \end{aligned}$$

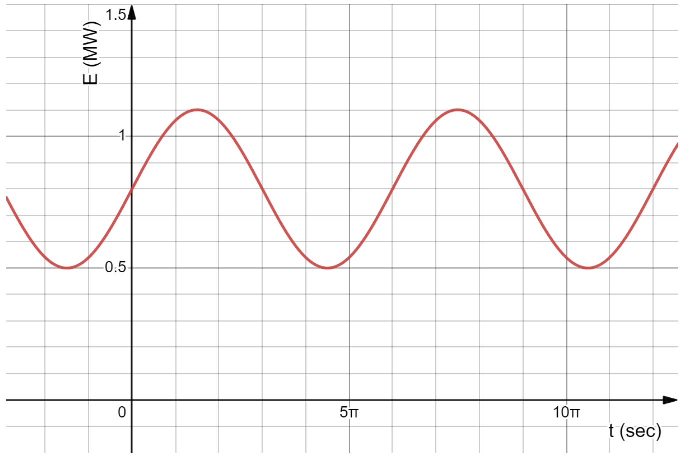
Such a number x exists for any value of y except $y = 2$. So any real number $y \neq 2$ is the range of f. In fact, if $y = 2$, we would get

$$2 = 2x + 5x - 3 \implies 2(x-3) = 2x+5 \implies 2x - 6 = 2x + 5 \implies -6 = 5,$$

which is obviously a contradiction. So $y = 2$ cannot be part of the range. Hence

$$R(f) = (-\infty, 2) \cup (2, \infty).$$

4. (10 pts) You have been hired as a math consultant for the next sequel to the movie Back to the Future. In this movie, Doc and Marty are trying to fix the DeLorean when they realize that the rectifier of flux capacitor is out of sync with the flux capacitor. Doc scans the flux capacitor with his scope and finds that the energy output (measured in gigawatts) as a function of time (measured in seconds) is a sinusoidal function, exactly as it should be. In order to tune the rectifier correctly, he needs to write down an expression for the function f whose graph is shown on the scope and is included below. This is where your math consulting skills come in. Starting with the graph of $g(x) = \sin(x)$, apply appropriate geometric transformations to g to find a formula for f so that its graph is the one shown below. Make sure you justify your answer.



The first thing to notice is the period of the function on this graph is 6π, while the period of $\sin(x)$ is 2π. So the graph of $\sin(x)$ needs to be stretched by a factor of 3. We know we can do that by dividing the input x by 3, as in $\sin(x/3)$.

Notice also that the distance between the peak and the trough of the wave in the graph is 0.6, while that distance for $\sin(x)$ is 2. So the graph of the sine function needs to be compressed by a factor of $0.6/2 = 0.3$, as in $0.3\sin(x)$.

Finally, notice that this graph passes through the point $(0, 0.8)$, while the function $0.3\sin(x)$ would pass through $(0, 0)$, just like $\sin(x)$ itself. Therefore $0.3\sin(x)$ needs to be shifted up by 0.8, as in $0.3\sin(x) + 0.8$.

We can now combine these three transformation. We have already combined the vertical compression and the shift, and we did it in the correct order, doing the compression before the shift. The horizontal stretch can be done in any order relative to these other two transformations. Here is the resulting function:

$$f(x) = 0.3\sin\left(\frac{x}{3}\right) + 0.8.$$

5. (5 pts each) **Extra credit problem.** Let

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

be a polynomial with leading coefficient $a_n \neq 0$.

(a) Show that $p(1/x)$ is a rational function $q(x)/r(x)$ where q and r are polynomials.

Hint: looking at an example may help you understand what is going on, but remember that one example does not prove that the statement is true for all polynomials.

Let us substitute $1/x$ into p:

$$\begin{aligned} p(x) &= a_n\left(\frac{1}{x}\right)^n + a_{n-1}\left(\frac{1}{x}\right)^{n-1} + \cdots + a_1\left(\frac{1}{x}\right) + a_0 \\ &= a_n\frac{1}{x^n} + a_{n-1}\frac{1}{x^{n-1}} + \cdots + a_1\frac{1}{x} + a_0 \end{aligned}$$

We can add the terms by bringing them to the common denominator x^n:

$$\begin{aligned} p(x) &= a_n\frac{1}{x^n} + a_{n-1}\frac{x}{x^n} + \cdots + a_1\frac{x^{n-1}}{x} + a_0\frac{x^n}{x^n} \\ &= \frac{a_n + a_{n-1}x + \cdots + a_1 x^{n-1} + a_0 x^n}{x^n} \end{aligned}$$

That looks like a rational function $q(x)/r(x)$ with the numerator and the denominator being polynomials:

$$q(x) = a_n + a_{n-1}x + \cdots + a_1 x^{n-1} + a_0 x^n$$
$$r(x) = x^n.$$

(b) Can you tell the degrees of q and r?

Hint: once again, looking some examples may be helpful in figuring out what a good general argument may be.

From the result in part (a), it is easy to see that the degree of r is n. The degree of q could be n as well, if $a_0 \neq 0$. But we do not know that $a_0 \neq 0$. In fact, it could be 0. So all we can say for sure is that $\deg(q) \leq n$. We do know that $a_n \neq 0$, so even if all the other coefficients of p are 0, the polynomial q has at least a nonzero constant term. Hence we can also tell that the degree of q is at least 0. This last observation just eliminates the possibility that if q were the 0 polynomial, its degree would not be a nonnegative integer. So we can conclude

$$0 \leq \deg(q) \leq n \quad \text{and} \quad \deg(r) = n.$$