MCS 118 Final Exam

1. (5 pts each) Is f a function from S to T? Why or why not? Do not forget to fully justify your answer.
   (a)

$$\begin{aligned} S &= \text{set of current Gustavus students} \\ T &= \text{set of dorms at GAC} \\ f(x) &= \text{the dorm } x \text{ lives in} \end{aligned}$$

   (b)

$$\begin{aligned} S &= \text{set of people who have siblings} \\ T &= \text{set of people who have ever lived} \\ f(x) &= x\text{'s oldest sibling} \end{aligned}$$

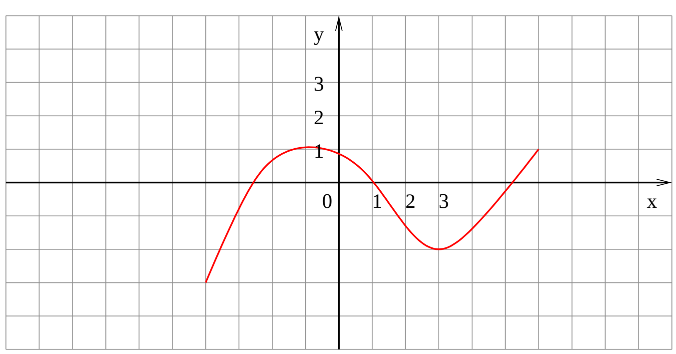
2. (5 pts each) Let f and g be functions of real numbers such that f can be composed with g. Suppose g is an odd function and let $h = f \circ g$.
   (a) Is h always an odd function? If you think it is, prove that it is; if you do not think it is, find a counterexample.
   (b) What if f is even? Is h always an odd or an even function then? Or can it be neither? If you think it is always odd or always even, prove that it is; if you do not think it is, find a counterexample.

3. (10 pts) Evaluate the limit if it exists:

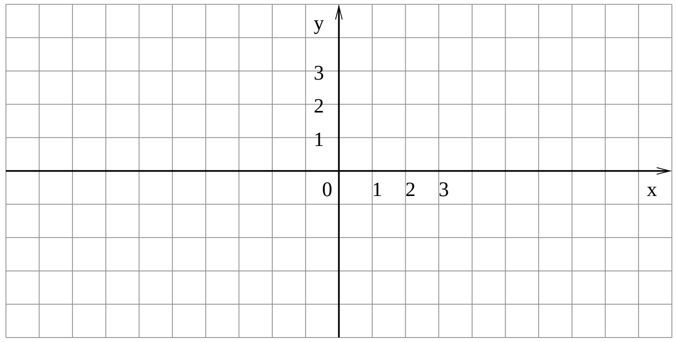
$$\lim_{x \to -1} \frac{x^2 + 2x + 1}{x^4 - 1}$$

   Do not forget to carefully justify each step in your calculation.

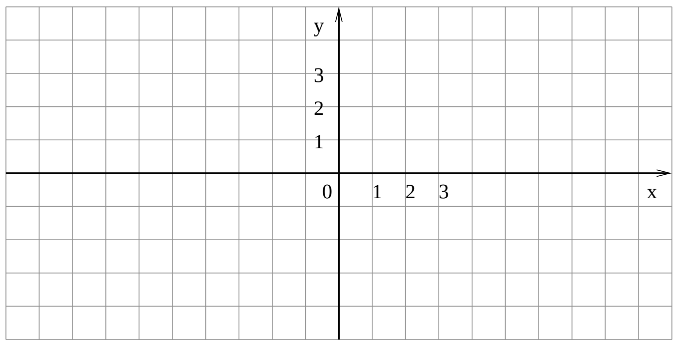
4. (5 pts each) The graph of a function $f(x)$ is given below.



   Sketch the graphs of the following functions. Make sure you explain your work.
   (a) $g(x) = -2f(x) - 3$



(b) $h(x) = f(2x - 3)$



5. (a) (4 pts) Let f be a function of real numbers and $a \in \mathbb{R}$ such that in some neighborhood of a, $f(x)$ has a value for x except possibly at $x = a$. State the formal (precise) definition of

$$\lim_{x \to a} f(x) = L$$

where L is a real number.

(b) (6 pts) Find

$$\lim_{x \to 5} |x - 5|^3.$$

You do not need to use the formal definition of the limit. Use whatever tools you need to find the limit, but be sure to give a rigorous argument to justify your work. (Hint: look at one-sided limits.)

6. (10 pts) Let $p(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$ be a polynomial function with real coefficients, that is $b_i \in \mathbb{R}$ for every $i = 0, 1 \ldots, n$. Prove that p has the direct substitution property

$$\lim_{x \to a} p(x) = p(a)$$

where a is any real number.

7. **Extra credit problem.** The purpose of this problem is to prove Limit Law 4: if f and g are functions of real numbers and $a \in \mathbb{R}$ such that

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

both exist, then

$$\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x).$$

I will break this down into steps.

(a) (10 pts) First, we will prove that the limit law holds in the special case when

$$\lim_{x\to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a} g(x) = 0.$$

So suppose

$$\lim_{x\to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a} g(x) = 0,$$

and use the formal definition of the limit (that is δ's and ϵ's) to prove that

$$\lim_{x\to a} [f(x)g(x)] = 0.$$

Hint: one way to do this is to find a good value of $\delta > 0$ such that if $0 < |x - a| < \delta$ then both $f(x)$ and $g(x)$ are closer to 0 than a distance of $\sqrt{\epsilon}$.

(b) (5 pts) Now, if

$$\lim_{x\to a} f(x) = L_1 \qquad \text{and} \qquad \lim_{x\to a} g(x) = L_2$$

then

$$\lim_{x\to a} (f(x) - L_1) = \lim_{x\to a} f(x) - \lim_{x\to a} L_1 = L_1 - L_1 = 0$$

and

$$\lim_{x\to a} (g(x) - L_2) = \lim_{x\to a} g(x) - \lim_{x\to a} L_2 = L_2 - L_2 = 0$$

by Limit Laws 2 and 7. So the functions $F(x) = f(x) - L_1$ and $G(x) = g(x) - L_2$ satisfy

$$\lim_{x\to a} F(x) = 0 \qquad \text{and} \qquad \lim_{x\to a} G(x) = 0.$$

Hence by part (a),

$$\lim_{x\to a} [F(x)G(x)] = 0.$$

Use that $f(x) = F(x) + L_1$ and $g(x) = G(x) + L_2$ and the limit laws (other than LL4) to show that

$$\lim_{x\to a} [f(x)g(x)] = L_1 L_2.$$

The Limit Laws

Let f and g are functions of real numbers and $a \in \mathbb{R}$ such that

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

both exist. Then the following are true:

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. if $\lim_{x\to a} g(x) \neq 0$, then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$
6. if $n \in \mathbb{Z}^+$, then $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$
7. for any $c \in \mathbb{R}$, $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. if $n \in \mathbb{Z}^+$, then $\lim_{x\to a} x^n = a^n$
10. if n is a positive odd number or if n is a positive even number and $a > 0$, then $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$
11. if n is a positive odd number or if n is a positive even number and $\lim_{x\to a} f(x) > 0$, then $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$