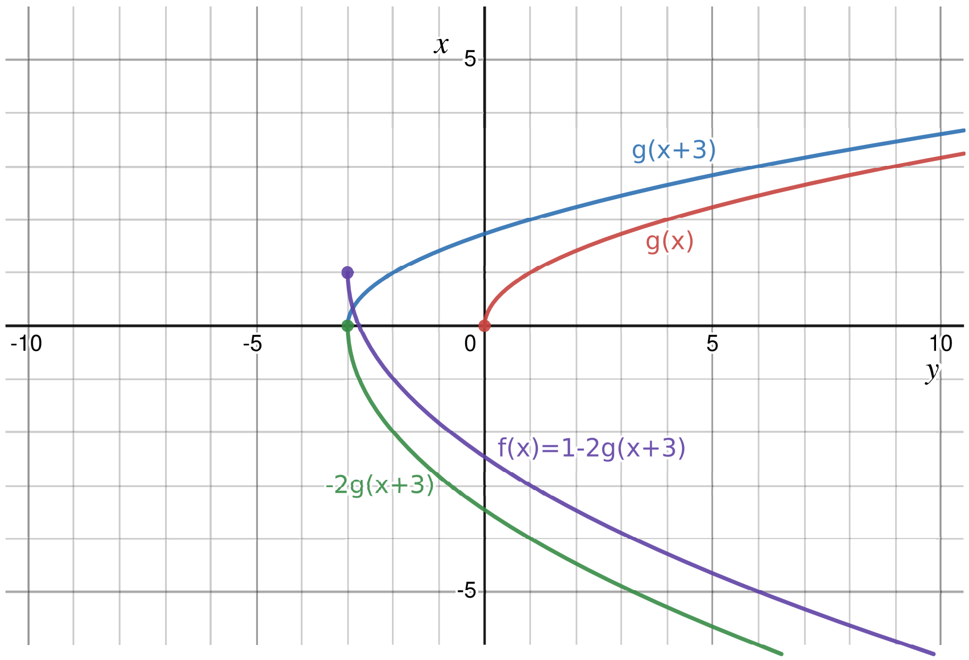
MCS 118 Final Exam Solutions

1. (10 pts) Graph the function $f(x) = 1 - 2\sqrt{x+3}$ by hand, not by plotting points, but by starting with the graph of a standard square root function and then applying appropriate transformations. Do not forget to explain what each transformation does to the graph.



Let $g(x) = \sqrt{x}$. The graph of g is familiar to all of us and is shown above. Then the graph of $g(x+3) = \sqrt{x+3}$ is the graph of g shifted to the left by 3 units. We now get the graph of $-2g(x+3) = -2\sqrt{x+3}$ by stretching the graph of $g(x+3)$ vertically by a factor of 2 and reflecting the result across the x-axis. Finally, we shift this graph up by 1 unit to get the graph of $f(x) = 1 - 2g(x+3) = 1 - 2\sqrt{x+3}$.

2. (5 pts each) Let $f(x) = \sqrt{3-x}$ and $g(x) = \sqrt{x^2-1}$.
   (a) Find the function $f - g$ and the largest possible subset of the real numbers that could be the domain of $f - g$.

   First,
   $$(f-g)(x) = f(x) - g(x) = \sqrt{3-x} - \sqrt{x^2-1}.$$
   For a real number x to be in the domain, both $f(x) = \sqrt{3-x}$ and $g(x) = \sqrt{x^2-1}$ must be real numbers. The value of $\sqrt{3-x}$ is real whenever
   $$3 - x \geq 0 \iff 3 \geq x.$$
   The value of $\sqrt{x^2-1}$ is real whenever
   $$x^2 - 1 \geq 0.$$
   Note that $h(x) = x^2 - 1$ is a quadratic function in which the x^2 term has a positive coefficient, so its graph is a right-side up parabola. Therefore its value is negative between its two roots. Since $x^2 - 1 = (x+1)(x-1)$, its two roots are 1 and -1. So $x^2 - 1 < 0$ when $-1 < x < 1$, and hence $x^2 - 1 \geq 0$ when $x \leq -1$ or $1 \leq x$.

Therefore $(f-g)(x) = \sqrt{3-x} - \sqrt{x^2-1}$ has a real value whenever $x \leq -1$ or $1 \leq x \leq 3$. So the largest possible domain of fg is

$$D(fg) = (\infty, -1] \cup [1, 3].$$

(b) Find the function f/g and the largest possible subset of the real numbers that could be the domain of f/g.

Now,

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{3-x}}{\sqrt{x^2-1}}.$$

For a real number x to be in the domain, both $f(x) = \sqrt{3-x}$ and $g(x) = \sqrt{x^2-1}$ must be real numbers and $g(x)$ cannot be 0. We already know from part (a) that $f(x)$ and $g(x)$ are real whenever $x \in (\infty, -1] \cup [1, 3]$, and also that

$$\sqrt{x^2-1} = 0 \iff x^2 - 1 = 0 \iff (x+1)(x-1) = 0 \iff x = -1 \text{ and } x = 1.$$

Therefore we cannot have -1 and 1 in the domain, and the largest possible domain is

$$D\,(fg) = (\infty, -1) \cup (1, 3].$$

3. (10 pts) Prove the statement using the $\delta - \epsilon$ definition of a limit:

$$\lim_{x \to -1.5} \frac{9 - 4x^2}{3 + 2x} = 6.$$

Make sure you carefully justify every step in your argument.

Hint: Simplify $\frac{9-4x^2}{3+2x}$, but be sure to explain why you can do that.

We need to show that for every $\epsilon > 0$ there is a corresponding $\delta > 0$ such that if

$$0 < |x - (-1.5)| < \delta$$

then

$$\left| \frac{9 - 4x^2}{3 + 2x} - 6 \right| < \epsilon.$$

So we want

$$\left| \frac{9 - 4x^2}{3 + 2x} - 6 \right| < \epsilon.$$

Now

$$\frac{9 - 4x^2}{3 + 2x} = \frac{(3-2x)(3+2x)}{3+2x} = 3 - 2x$$

unless $3 + 2x = 0$, which would be the case if $x = -1.5$. But we know $0 < |x - (-1.5)| = |x + 1.5|$, so $x + 1.5$ cannot be 0. Hence $2x + 3 = 2(x + 1.5) \neq 0$.

Therefore, for the values of x under consideration,

$$\left| \frac{9 - 4x^2}{3 + 2x} - 6 \right| = |3 - 2x - 6| = |-2x - 3| = |2x + 3| = |2||x + 1.5| = 2|x - (-1.5)|.$$

We want to make this less than ϵ. So let us choose $\delta = \epsilon/2$. Now, we will show that if

$$0 < |x - (-1.5)| < \delta = \frac{\epsilon}{2}$$

then

$$\left| \frac{9 - 4x^2}{3 + 2x} - 6 \right| < \epsilon.$$

Suppose

$$0 < |x - (-1.5)| < \frac{\epsilon}{2}.$$

Then we know $x-(-1.5)\neq 0$, and so $2x+3=2(x+1.5)\neq 0$. Now,

$$\begin{aligned}
\left|\frac{9-4x^2}{3+2x}-6\right| &= \left|\frac{(3-2x)(3+2x)}{3+2x}-6\right| \\
&= |3-2x-6| \\
&= |-2x-3| \\
&= |2x+3| \\
&= |2||x+1.5| \\
&= 2|x-(-1.5)| \\
&< 2\frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

which is exactly what we wanted to show.

4. (a) (3 pts) Let f be a function of real numbers whose domain includes the interval I. Define what it means for f to be a decreasing function.

   The function f is decreasing on I if for any x_1, x_2 in I, if $x_1 < x_2$ then $f(x_1) > f(x_2)$.

   (b) (7 pts) Show that the function $f(x)=\frac{1}{x-2}$ is decreasing on the interval $(2,\infty)$ by proving that it satisfies the definition.

   Let $x_1, x_2 \in (2,\infty)$ such that $x_1 < x_2$. Then

   $$x_1 < x_2 \implies x_1 - 2 < x_2 - 2.$$

   It is clear that $x_1 - 2 > 0$ and $x_2 - 2 > 0$. Therefore we can divide both sides of

   $$x_1 - 2 < x_2 - 2$$

   by the positive number $x_2 - 2$ to get

   $$\frac{x_1-2}{x_2-2} < 1.$$

   Dividing both sides by the positive number $x_1 - 2$ results in

   $$\frac{1}{x_2-2} < \frac{1}{x_1-2}.$$

   Therefore

   $$f(x_1) = \frac{1}{x_1-2} > \frac{1}{x_2-2} = f(x_2).$$

5. (10 pts) Use the limit laws to prove that the function

$$f(x) = \frac{3x^2+x+4}{\sqrt{x^2+3}}$$

is continuous at every real number a.

We need to show

$$\lim_{x\to a} f(x) = f(a).$$

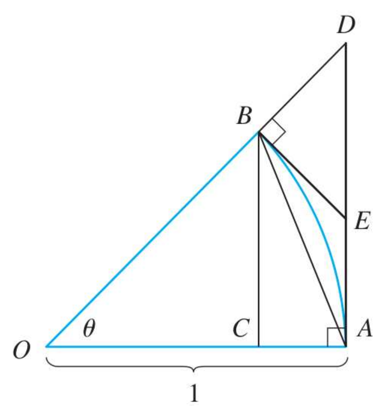
Let us calculate the limit:

$$\begin{aligned}
\lim_{x\to a} \frac{3x^2+x+4}{\sqrt{x^2+3}} &= \frac{\lim_{x\to a}(3x^2+x+4)}{\lim_{x\to a}\sqrt{x^2+3}} && \text{by LL5}\\
&= \frac{\lim_{x\to a}(3x^2)+\lim_{x\to a}x+\lim_{x\to a}4}{\sqrt{\lim_{x\to a}(x^2+3)}} && \text{by LL1 and LL1}\\
&= \frac{3\lim_{x\to a}(x^2)+a+4}{\sqrt{\lim_{x\to a}(x^2+3)}} && \text{by LL3, LL8 and LL7}\\
&= \frac{2a^2+a+4}{\sqrt{\lim_{x\to a}x^2+\lim_{x\to a}3}} && \text{by LL9 and LL1}\\
&= \frac{2a^2+a+4}{\sqrt{a^2+3}} && \text{by LL9 and LL7}\\
&= f(a)
\end{aligned}$$

Notice that we were justified in using LL11 because the quantity under the square root turns out to approach a^2+3, which is positive for every real number a. Hence f is continuous at every $a \in \mathbb{R}$.

6. (a) (7 pts) Use the Squeeze Theorem and the diagram below to show that

$$\lim_{x\to 0^+} \cos(x) = 1.$$



We will start by noting that

$$0 < AC < AB$$

because both AC and AB are the sides of the right triangle $\triangle ABC$, but AB is the hypotenuse, so it is the longest side of $\triangle ABC$. Then

$$AB < \widehat{AB} = \theta$$

because AB is a straight line segment from A to B and is therefore shorter than the curved arc $\widehat{AB}$. Now,

$$\cos(\theta) = OC = OA - AC = 1 - AC.$$

Since

$$0 < AC < AB < \widehat{AB} \implies 0 > -AC > -AB > -\widehat{AB}$$

we can now add 1 to the sides of the last inequality to obtain

$$1-0>\underbrace{1-AC}_{\cos(\theta)}>1-AB>1-\widehat{AB}=1-\theta.$$

Hence

$$1>\cos(\theta)>1-\theta.$$

Note that

$$\lim_{\theta\to 0^+} 1 = 1 \qquad \text{by LL7}$$

$$\lim_{\theta\to 0^+} (1-\theta) = 1 \qquad \text{by direct substitution into the polynomial.}$$

By the Squeeze Theorem,

$$\lim_{\theta\to 0^+} \cos(\theta) = 1,$$

(b) (3 pts) Now use the fact that cosine is an even function to figure out

$$\lim_{x\to 0^-} \cos(x)$$

and use this to show that

$$\lim_{x\to 0} \cos(x) = 1.$$

Since cosine is even, $\cos(-x)=\cos(x)$. If $x\to 0^-$, then x is a negative number. Let $x=-y$ where y is positive. It is quite clear that $x\to 0^-$ if and only if $y\to 0^+$. Hence

$$\lim_{x\to 0^-} \cos(x) = \lim_{y\to 0^+} \cos(y) = 1$$

by part (a).
Since both the left and the right limits as $x\to 0^-$ and as $x\to 0^+$ of $\cos(x)$ are 1,

$$\lim_{x\to 0^+} \cos(x) = 1.$$

7. **Extra credit problem.** Remember that a rational number is a number of the form m/n where m and n are integers and $n\neq 0$, and an irrational number is a real number that cannot be expressed as a ratio of two integers. For example, $2/3$ and $-13/5$ are rational numbers, but π and $\sqrt{2}$ are known to be irrational. One property (not essential for answering the question below) of rational/irrational numbers you may have encountered is that rational numbers have finite or infinite but repeating decimal expansions, whereas irrational numbers have infinite decimal expansions without a repeating pattern. One important property of the real numbers–and this is relevant to the question below–is that any open interval (a,b), however small, contains both rational and irrational numbers.

Let $f:\mathbb{R}\to\mathbb{R}$ be the function

$$f(x)=\begin{cases} x & \text{if } x \text{ is a rational number} \\ x^2 & \text{if } x \text{ is an irrational number.}\end{cases}$$

(a) (10 pts) Find

$$\lim_{x\to 0^+} f(x)$$

and carefully justify your answer.

Hint: You can do this either by using the Squeeze Theorem or by a $\delta-\epsilon$ argument.

Here is the Squeeze Theorem argument: Note that for $0<x<1$, we have

$$0<x<1 \implies 0<x^2<x$$

by multiplying all sides by the positive number x. Since $f(x)$ is either equal to the smaller number x^2 or to the larger number x,

$$x^2 \leq f(x) \leq x$$

for all $x \in (0,1)$. So we can use $g(x) = x^2$ and $h(x) = x$ as the squeezing functions. The limits of g and h are

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} x^2 = 0 \qquad \text{by LL 9}$$
$$\lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} x = 0 \qquad \text{by LL 8.}$$

By the Squeeze Theorem,

$$\lim_{x\to 0^+} f(x) = 0.$$

Here is the $\delta - \epsilon$ argument: We want to prove that for every $\epsilon > 0$ there is a corresponding $\delta > 0$ such that if $0 < x < \delta$, then $|f(x) - 0| < \epsilon$. Since $f(x)$ is either x or x^2, we want x to be close enough to 0 that

$$|x - 0| < \epsilon \qquad \text{and} \qquad |x^2 - 0| < \epsilon$$

are both true. Notice that since x is a positive number, so is x^2. Therefore

$$|x - 0| = |x| = x \qquad \text{and} \qquad |x^2 - 0| = |x^2| = x^2.$$

So we want

$$x < \epsilon \qquad \text{and} \qquad x^2 < \epsilon.$$

In fact, just like above, we know that for $x \in (0,1)$,

$$0 < x < 1 \implies 0 < x^2 < x.$$

So if we make $x < \epsilon$, then we have

$$0 < x^2 < x < \epsilon.$$

Of course, this assumes that $x \in (0,1)$. So we also need to make sure $0 < x < 1$ just in case $\epsilon > 1$. Therefore we will set $\delta - = \min(\epsilon, 1)$.
We will now show that if $0 < x < \delta = \min(\epsilon, 1)$, then $|f(x) - 0| < \epsilon$. Suppose $0 < x < \delta$. Then

$$0 < x < \delta \leq 1 \qquad \text{and} \qquad 0 < x < \delta \leq \epsilon.$$

It follows from the first of these inequalities, just like above, that

$$0 < x^2 < x.$$

Using the second inequality now gives

$$0 < x^2 < x < \epsilon.$$

Hence

$$0 < f(x) < \epsilon \implies |f(x)| < \epsilon \implies |f(x) - 0| < \epsilon.$$

(b) (5 pts) Does the left limit

$$\lim_{x\to 0^-} f(x)$$

also exist? If so, find its value; if it does not, explain why it does not exist.

Hint: How is this similar to part (a)? How is it different?

Yes, it does and it is also 0. We can show this much the same way as in part (a).
Here is the Squeeze Theorem argument: Note that for $-1 < x < 0$, we have

$$-1 < x < 0 \implies 0 < x^2 < -x$$

by multiplying all sides by the negative number x. Since $f(x)$ is either equal to the positive number x^2 or to the negative number x,

$$x \leq f(x) \leq x^2$$

for all $x \in (0,1)$. So we can use $g(x) = x$ and $h(x) = x^2$ as the squeezing functions. The limits of g and h are

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} x = 0 \qquad \text{by LL 8}$$

$$\lim_{x \to 0^-} h(x) = \lim_{x \to 0^-} x^2 = 0 \qquad \text{by LL 9.}$$

By the Squeeze Theorem,

$$\lim_{x \to 0^-} f(x) = 0.$$

Here is the $\delta - \epsilon$ argument: We want to prove that for every $\epsilon > 0$ there is a corresponding $\delta > 0$ such that if $-\delta < x < 0$, then $|f(x) - 0| < \epsilon$. Since $f(x)$ is either x or x^2, we want x to be close enough to 0 that

$$|x - 0| < \epsilon \qquad \text{and} \qquad |x^2 - 0| < \epsilon$$

are both true. Since $|x - 0| = |x|$ and $|x^2 - 0| = |x^2| = |x|^2$, we really want

$$|x| < \epsilon \qquad \text{and} \qquad |x|^2 < \epsilon.$$

Since x is a negative number close to 0, it is not too much to ask that x should be between -1 and 0, by setting $\delta = 1$ for now. Then

$$-1 < x < 0 \implies 0 < |x| < 1 \implies 0 < |x|^2 < |x|$$

where we got the third inequality from the second by multiplying all sides by the positive number $|x|$. So if we make $|x| < \epsilon$, then we have

$$|x|^2 < |x| < \epsilon.$$

Since we also want $-1 < x < 0$, we will set $\delta = \min(\epsilon, 1)$.
We will now show that if $-\delta < x < 0$, then $|f(x) - 0| < \epsilon$. So suppose $-\delta < x < 0$. Since $\delta = \min(\epsilon, 1)$, either $\delta = \epsilon$ or $\delta = 1$, whichever is smaller, and so

$$\delta \leq \epsilon \qquad \text{and} \qquad \delta \leq 1$$

are both true. Multiplying the latter by -1 gives $-\delta \geq -1$. Therefore we know

$$-1 \leq -\delta < x < 0.$$

Just like above, this implies

$$0 < |x| < 1 \implies 0 < |x|^2 < |x|.$$

Hence

$$|x^2| = |x|^2 < |x| < \delta \leq \epsilon.$$

Since $|f(x) - 0| = |f(x)|$ which is either $|x|$ or $|x^2|$, we can conclude that

$$|f(x) - 0| < \epsilon.$$