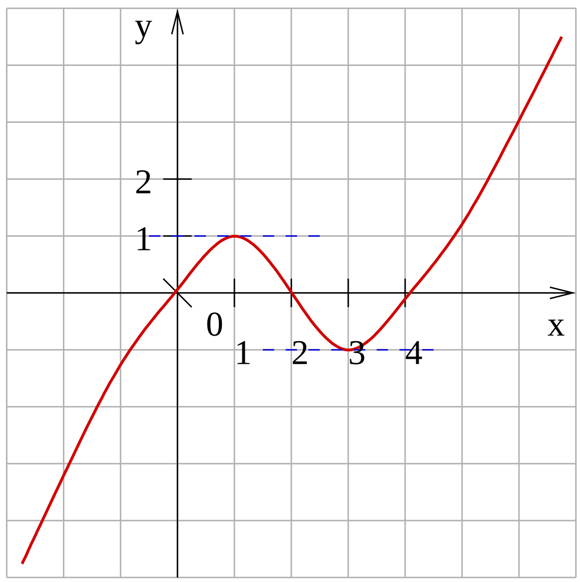
MCS 119 Exam 1

1. (10 pts) Sketch the graph of a function g for which
   - $g(0) = g(2) = g(4) = 0$
   - $g'(1) = g'(3) = 0$
   - $g'(0) = g'(4) = 1$
   - $g'(2) = -1$
   - $\lim_{x\to\infty} g(x) = \infty$
   - $\lim_{x\to-\infty} g(x) = -\infty$

   Do not forget to justify your work, for example, by explaining how you used each condition in constructing your graph.



We are given $g(0) = g(2) = g(4) = 0$, which tells us that the graph must pass through the points $(0,0)$, $(2,0)$, $(4,0)$ on the x-axis. To satisfy $g'(0) = g'(4) = 1$, the graph must be increasing as it passes through $(0,0)$ and $(4,0)$, and it must actually be increasing at a rate of 1, which tells us how steep it at these two points. Similarly, the graph must be sloping down at a slope of -1 as it passes through $(2,0)$. The condition $g'(1) = g'(3) = 0$ tells us that the tangent lines are horizontal at $x = 1$ and at $x = 3$, which is why the graph has a peak and a valley at these values of x. Finally, the two limits tell us that as x increases toward ∞, the graph is headed toward the upper right, and as x decreases toward $-\infty$, the graph is headed to the lower left.

2. (10 pts) Use the definition of the derivative to find the derivative of $f(x) = \sqrt{6-x}$.

$$
\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x-a} \\
&= \lim_{x\to a} \frac{\sqrt{6-x} - \sqrt{6-a}}{x-a} \\
&= \lim_{x\to a} \left[\frac{\sqrt{6-x} - \sqrt{6-a}}{x-a} \, \frac{\sqrt{6-x} + \sqrt{6-a}}{\sqrt{6-x} + \sqrt{6-a}} \right]
\end{aligned}
$$

Notice that we can multiply and divide by $\sqrt{6-x}+\sqrt{6-a}$ because it cannot be 0. Since $\sqrt{6-x} \geq 0$ and $\sqrt{6-a} \geq 0$, the only way $\sqrt{6-x}+\sqrt{6-a}$ could 0 is if both $\sqrt{6-x}=0$ and $\sqrt{6-a}=0$, which happens only if $x=a=6$. But $x \to a$ means x is approaching a but $x \neq a$. Let us just work on the expression in the limit:

$$\begin{aligned}
\frac{\sqrt{6-x}-\sqrt{6-a}}{x-a}\,\frac{\sqrt{6-x}+\sqrt{6-a}}{\sqrt{6-x}+\sqrt{6-a}} &= \frac{\sqrt{6-x}^2-\sqrt{6-a}^2}{(x-a)(\sqrt{6-x}+\sqrt{6-a})}\\
&= \frac{6-x-(6-a)}{(x-a)(\sqrt{6-x}+\sqrt{6-a})}\\
&= \frac{a-x}{(x-a)(\sqrt{6-x}+\sqrt{6-a})}\\
&= \frac{-(x-a)}{(x-a)(\sqrt{6-x}+\sqrt{6-a})}
\end{aligned}$$

We can now cancel $x-a$ as it is not 0. Hence

$$\begin{aligned}
f'(a) &= \lim_{x\to a}\frac{-1}{\sqrt{6-x}+\sqrt{6-a}}\\
&= \frac{\lim_{x\to a}(-1)}{\lim_{x\to a}\left(\sqrt{6-x}+\sqrt{6-a}\right)} && \text{by LL5}\\
&= \frac{\lim_{x\to a}(-1)}{\lim_{x\to a}\sqrt{6-x}+\lim_{x\to a}\sqrt{6-a}} && \text{by LL1}\\
&= \frac{-1}{\sqrt{\lim_{x\to a}(6-x)}+\sqrt{6-a}} && \text{by LL7 and LL11}\\
&= \frac{-1}{\sqrt{6-a}+\sqrt{6-a}} && \text{by direct substitution}\\
&= -\frac{1}{2\sqrt{6-a}}
\end{aligned}$$

So

$$f'(x) = \frac{-1}{2\sqrt{6-x}}.$$

Alternately, we could have argued that

$$\lim_{x\to a}\frac{-1}{\sqrt{6-x}+\sqrt{6-a}}$$

can be evaluated by direct substitution because the function $g(x)=\frac{-1}{\sqrt{6-x}+\sqrt{6-a}}$ is continuous. This is because we know $6-x$ is continuous, as it is a polynomial and so $\sqrt{6-x}$ is also continuous at every x in its domain by Theorem 6 in Section 1.5. Now, -1 and $\sqrt{6-a}$ do not depend on x, so they are constant functions and therefore continuous. Finally, sums and quotients of continuous functions are continuous by Theorem 4 in 1.6.

3. (5 pts each) Let

$$f(x) = 5\,\frac{x^2+3x-10}{x^2-3x+2}.$$

(a) Use the definition of vertical asymptote to find the vertical asymptote(s) of $f(x)$.

By definition, f has a vertical asymptote is at $x=a$ if one of its one-sided limits at a is ∞ or $-\infty$. Since f is a rational function, the only reason it may have such a limit is if the denominator approaches 0 at a, while the numerator approaches a nonzero number. To figure out where this may happen, we will factor the numerator and the denominator:

$$f(x) = 5\frac{(x+5)(x-2)}{(x-2)(x-1)} = \frac{5(x+5)(x-2)}{(x-2)(x-1)}$$

Notice that the denominator is 0 at $x = 1$ and $x = 2$. But the numerator is also 0 at $x = 2$.

Let us see what happens if $x \to 1^+$. First, notice that we can cancel $x-2$ since $x-2 \neq 0$ when x is a number getting close to 1. So

$$f(x) = \frac{5(x+5)}{x-1}.$$

As $x \to 1^+$, $x - 1$ approaches 0, while $x - 1 > 0$. Hence the denominator is a positive number that is getting closer and closer to 0. The numerator on the other hand is approaching $5(1 + 5) = 30$ (by direct substitution or by using the limit laws). So the value of f is a large positive number and is becoming larger and larger. Therefore

$$\lim_{x\to 1^+} f(x) = \infty.$$

This shows $f(x)$ has a vertical asymptote at $x = 1$.

What about $x = 2$? Note that

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{5(x+5)(x-2)}{(x-2)(x-1)} = \lim_{x\to 2} \frac{5(x+5)}{x-1}$$

because we can again cancel $x - 2$, as $x - 2 \neq 0$ when $x \to 2$. The expression in the limit is a rational function, so the limit can be evaluated by direct substitution if this does not result in 0 in the denominator:

$$\lim_{x\to 2} \frac{5(x+5)}{x-1} = \frac{5(2+5)}{2-1} = 35.$$

Since the limit is 35, the left and right limits must be 35 as well. Hence $x = 2$ is not a vertical asymptote.

So $x = 1$ is the only vertical asymptote of $f(x)$.

(b) Use the definition of horizontal asymptote to find the horizontal asymptote(s) of $f(x)$.

$$\begin{aligned}
\lim_{x\to\infty} \left[5\,\frac{x^2+3x-10}{x^2-3x+2}\right] &= \lim_{x\to\infty} \left[5\,\frac{\frac{x^2}{x^2}+\frac{3x}{x^2}-\frac{10}{x^2}}{\frac{x^2}{x^2}-\frac{3x}{x^2}+\frac{2}{x^2}}\right] \\
&= \lim_{x\to\infty} \left[5\,\frac{1+\frac{3}{x}-\frac{10}{x^2}}{1-\frac{3}{x^2}+\frac{2}{x^2}}\right] \\
&= 5\,\frac{\lim_{x\to\infty}\left(1+\frac{3}{x}-\frac{10}{x^2}\right)}{\lim_{x\to\infty}\left(1-\frac{3}{x^2}+\frac{2}{x^2}\right)} && \text{by LL\# 3 and LL\# 5} \\
&= 5\,\frac{1}{1} \\
&= 5
\end{aligned}$$

where we used the fact that $\frac{3}{x}$, $\frac{10}{x^2}$, and $\frac{2}{x^2}$ all approach 0 as $x \to \infty$. Similarly,

$$\begin{aligned}
\lim_{x\to-\infty}\left[5\,\frac{x^2+3x-10}{x^2-3x+2}\right] &= \lim_{x\to-\infty}\left[5\,\frac{\frac{x^2}{x^2}+\frac{3x}{x^2}-\frac{10}{x^2}}{\frac{x^2}{x^2}-\frac{3x}{x^2}+\frac{2}{x^2}}\right] \\
&= \lim_{x\to-\infty}\left[5\,\frac{1+\frac{3}{x}-\frac{10}{x^2}}{1-\frac{3}{x^2}+\frac{2}{x^2}}\right] \\
&= 5\,\frac{1}{1} \\
&= 5
\end{aligned}$$

Therefore the only horizontal asymptote of $f(x)$ is $y = 5$.

4. (a) (6 pts) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a constant function $f(x) = c$ then $f'(x) = 0$.

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{c-c}{x-a} = \lim_{x\to a}\frac{0}{x-a} = \lim_{x\to a} 0 = 0,$$

where we used the fact that $x - a \neq 0$ as $x \to a$, so $\frac{0}{x-a} = 0$.

(b) (4 pts) Use the Power Rule to differentiate $f(x) = 1/\sqrt[7]{x^2}$.

$$\frac{d}{dx}1/\sqrt[7]{x^2} = \frac{d}{dx}x^{-2/7} = -\frac{2}{7}x^{-2/7-1} = -\frac{2}{7}x^{-9/7} = -\frac{2}{7\sqrt[7]{x^9}}$$

5. (10 pts) **Extra credit problem.** Let n be a positive integer and let $f(x) = \sqrt[n]{x} = x^{1/n}$. Use the definition of the derivative to find $f'(x)$. Hint: You can do this similarly to how we found the derivative of x^n in class. The algebraic identity

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + a^2b^{n-3} + ab^{n-2} + b^{n-1})$$

may be useful for factoring the denominator.

$$\begin{aligned}
f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a}\frac{\sqrt[n]{x}-\sqrt[n]{a}}{x-a}
\end{aligned}$$

Now, we can factor $x - a$ as

$$\begin{aligned}
x - a &= \sqrt[n]{x}^n - \sqrt[n]{a}^n \\
&= \big(\sqrt[n]{x} - \sqrt[n]{a}\big)\big(\sqrt[n]{x}^{n-1} + \sqrt[n]{x}^{n-2}\sqrt[n]{a} + \cdots + \sqrt[n]{x}\,\sqrt[n]{a}^{n-2} + \sqrt[n]{a}^{n-1}\big)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\sqrt[n]{x}-\sqrt[n]{a}}{x-a} &= \frac{\sqrt[n]{x}-\sqrt[n]{a}}{\big(\sqrt[n]{x} - \sqrt[n]{a}\big)\big(\sqrt[n]{x}^{n-1} + \sqrt[n]{x}^{n-2}\sqrt[n]{a} + \cdots + \sqrt[n]{x}\,\sqrt[n]{a}^{n-2} + \sqrt[n]{a}^{n-1}\big)} \\
&= \frac{1}{\sqrt[n]{x}^{n-1} + \sqrt[n]{x}^{n-2}\sqrt[n]{a} + \cdots + \sqrt[n]{x}\,\sqrt[n]{a}^{n-2} + \sqrt[n]{a}^{n-1}}
\end{aligned}$$

where we could cancel $\sqrt[n]{x} - \sqrt[n]{a}$ because $x \neq a$ as $x \to a$, so $\sqrt[n]{x} - \sqrt[n]{a} \neq 0$. Now, we will have to do the limit as $x \to a$. Notice that

$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} \qquad \text{by LL\#10}$$

$$\lim_{x \to a} \sqrt[n]{x}^k = \sqrt[n]{a}^k \qquad \text{by LL\#6}$$

$$\lim_{x \to a} \sqrt[n]{x}^k \sqrt[n]{a}^{n-1-k} = \sqrt[n]{a}^k \sqrt[n]{a}^{n-1-k} = \sqrt[n]{a}^{n-1} \qquad \text{by LL\#3 and LL\#6}$$

Therefore

$$\lim_{x \to a} \left(\sqrt[n]{x}^{n-1} + \sqrt[n]{x}^{n-2} \sqrt[n]{a} + \cdots + \sqrt[n]{x} \sqrt[n]{a}^{n-2} + \sqrt[n]{a}^{n-1} \right) = n \sqrt[n]{a}^{n-1}$$

by LL#1 and the fact that the sum

$$\sqrt[n]{x}^{n-1} + \sqrt[n]{x}^{n-2} \sqrt[n]{a} + \cdots + \sqrt[n]{x} \sqrt[n]{a}^{n-2} + \sqrt[n]{a}^{n-1}$$

has n terms.

Now, by Limit Laws #5 and #7,

$$f'(a) = \lim_{x \to a} \frac{1}{\sqrt[n]{x}^{n-1} + \sqrt[n]{x}^{n-2} \sqrt[n]{a} + \cdots + \sqrt[n]{x} \sqrt[n]{a}^{n-2} + \sqrt[n]{a}^{n-1}} = \frac{1}{n \sqrt[n]{a}^{n-1}}.$$