MCS 119 Exam 2 Solutions

1. (10 pts) Find an equation of the tangent line to the curve $y = x\sqrt{x}$ that is parallel to the line $y = 1 + 3x$.

To be parallel to $y = 1 + 3x$, the two lines must have the same slope. So we are looking for the value of x where the derivative of the curve is 3:

$$3 = \frac{dy}{dx} = \frac{d}{dx}x\sqrt{x} = \frac{d}{dx}x^{3/2} = \frac{3}{2}x^{1/2}.$$

Now

$$\frac{3}{2}x^{1/2} = 3 \implies x^{1/2} = \frac{2}{3}3 = 2 \implies \sqrt{x} = 2 \implies x = 4.$$

When $x = 4$, $y = 4\sqrt{4} = 8$. So the tangent line has slope 3, and passes through the point $(4, 8)$. Therefore its equation is $y = 3x + b$ where b satisfies $8 = 3(4) + b$, so $b = -4$. Hence the tangent line in question is $y = 3x - 4$.

2. (5 pts each)
   (a) If f is a differentiable function, find an expression for the derivative of

$$y = \frac{1 + xf(x)}{\sqrt{x}}.$$

By the Quotient Rule and the Product Rule,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{d}{dx}\big(1 + xf(x)\big)\sqrt{x} - \big(1 + xf(x)\big)\frac{d}{dx}\sqrt{x}}{\sqrt{x}^2} \\
&= \frac{\big(\frac{d}{dx}x\, f(x) + x\frac{df}{dx}\big)\sqrt{x} - \big(1 + xf(x)\big)\frac{d}{dx}x^{1/2}}{x} \\
&= \frac{\big(f(x) + xf'(x)\big)\sqrt{x} - \big(1 + xf(x)\big)\frac{1}{2}x^{-1/2}}{x} \\
&= \frac{\big(f(x) + xf'(x)\big)\sqrt{x} - \big(1 + xf(x)\big)\frac{1}{2\sqrt{x}}}{x} \\
&= \frac{2x\big(f(x) + xf'(x)\big) - \big(1 + xf(x)\big)}{2x\sqrt{x}} \\
&= \frac{2x(f(x) + 2x^2 f'(x) - 1 - xf(x)}{2x\sqrt{x}} \\
&= \frac{x(f(x) + 2x^2 f'(x) - 1}{2x\sqrt{x}}
\end{aligned}$$

   (b) Write $|x| = \sqrt{x^2}$ and use the Chain Rule to show that

$$\frac{d}{dx}|x| = \frac{x}{|x|}.$$

$$\begin{aligned}
\frac{d}{dx}|x| &= \frac{d}{dx}\sqrt{x^2} \\
&= \frac{d}{dx}\left(x^2\right)^{1/2} \\
&= \frac{1}{2}\left(x^2\right)^{-1/2}\frac{d}{dx}x^2 \\
&= \frac{1}{2\sqrt{x^2}}\,2x \\
&= \frac{x}{|x|}
\end{aligned}$$

3. (10 pts) Use the definition of the derivative to prove the Product Rule. (Hint: Express $f(x+h)$ as $f(x)+\Delta f$ where $\Delta f = f(x+h) - f(x)$, or express $f(x)$ as $f(a)+\Delta f$ where $\Delta f = f(x) - f(a)$, depending on which version of the definition of the derivative you are using.)

We want to prove that if $f(x)$ and $g(x)$ are both differentiable functions at $x = a$ then

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

Here is the argument we gave in class. Or see p. 107 in your textbook.

By the definition of the derivative,

$$(fg)'(a) = \lim_{x\to a}\frac{f(x)g(x) - f(a)g(a)}{x-a}.$$

Now, let $\Delta f = f(x) - f(a)$ and $\Delta g = g(x) - g(a)$ and substitute $f(x) = f(a) + \Delta f$ and $g(x) = g(a) + \Delta g$:

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a} &= \lim_{x\to a}\frac{(f(a)+\Delta f)(g(a)+\Delta g) - f(a)g(a)}{x-a} \\
&= \lim_{x\to a}\frac{f(a)g(a)+\Delta f g(a) + f(a)\Delta g + \Delta f\Delta g - f(a)g(a)}{x-a} \\
&= \lim_{x\to a}\frac{\Delta f g(a) + f(a)\Delta g + \Delta f\Delta g}{x-a} \\
&= \lim_{x\to a}\frac{\Delta f g(a)}{x-a} + \lim_{x\to a}\frac{f(a)\Delta g}{x-a} + \lim_{x\to a}\frac{\Delta f\Delta g}{x-a}.
\end{aligned}$$

Now, since $g(a)$ does not depend on x, it is constant factor and by Limit Law #3,

$$\begin{aligned}
\lim_{x\to a}\frac{\Delta f g(a)}{x-a} &= \mathfrak{g}(a)\lim_{x\to a}\frac{\Delta f}{x-a} \\
&= \mathfrak{g}(a)\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \\
&= g(a)f'(a).
\end{aligned}$$

Similarly,

$$\lim_{x\to a}\frac{f(a)\Delta g}{x-a} = f(a)g'(a).$$

Finally, by Limit Law #4,

$$\lim_{x\to a}\frac{\Delta f\Delta g}{x-a} = \left[\lim_{x\to a}\Delta f\right]\left[\lim_{x\to a}\frac{\Delta g}{x-a}\right].$$

Of this

$$\lim_{x \to a} \frac{\Delta g}{x - a} = g'(a),$$

while

$$\begin{aligned}
\lim_{x \to a} \Delta f &= \lim_{x \to a} \big(f(x) - f(a)\big) \\
&= \lim_{x \to a} f(x) - \lim_{x \to a} f(a) && \text{by LL\#2} \\
&= \lim_{x \to a} f(x) - f(a) && \text{by LL\#7}
\end{aligned}$$

Since $f(x)$ is a differentiable at $x = a$, it must also be continuous there. Hence $\lim_{x \to a} f(x) = f(a)$. Now

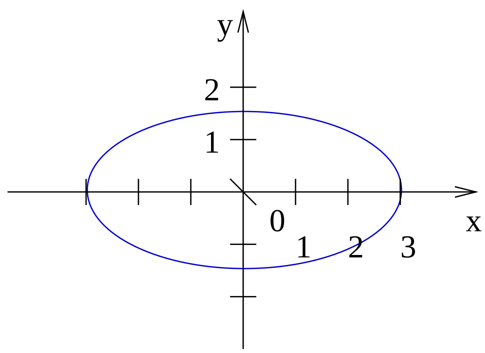
$$\lim_{x \to a} f(x) - f(a) = f(a) - f(a) = 0.$$

Putting the pieces together, we have obtained

$$(fg)'(a) = g(a)f'(a) + f(a)g'(a),$$

which is what we wanted to show.

4. (10 pts) The equation $x^2 + 4y^2 = 9$ describes the ellipse shown in the diagram below.



Find the equation of the tangent line to this ellipse at the point $(1, -\sqrt{2})$.

We can do this by implicit differentiaton, treating y as a function $y(x)$. First, we differentiate both sides of the equation

$$\begin{aligned}
\frac{d}{dx}(x^2 + 4y^2) &= \frac{d}{dx} 9 \\
2x + 4(2y)\frac{dy}{dx} &= 0
\end{aligned}$$

We can now solve this linear equation for $\frac{dy}{dx}$:

$$\begin{aligned}
2x + 4(2y)\frac{dy}{dx} &= 0 \\
8y\frac{dy}{dx} &= -2x \\
\frac{dy}{dx} &= \frac{-2x}{8y} = -\frac{x}{4y}
\end{aligned}$$

At $(x, y) = (1, -\sqrt{2})$:

$$\frac{dy}{dx} = -\frac{1}{4(-\sqrt{2})} = \frac{1}{4\sqrt{2}}.$$

This is the slope of the tangent line. So its equation is $y = \frac{1}{4\sqrt{2}}x + b$. The line passes through $(1, -\sqrt{2})$, so

$$-\sqrt{2} = \frac{1}{4\sqrt{2}}1 + b \implies b = -\sqrt{2} - \frac{1}{4\sqrt{2}} = -\frac{4\sqrt{2}\sqrt{2}+1}{4\sqrt{2}} = -\frac{9}{4\sqrt{2}}.$$

So the equation of the tangent line is

$$y = \frac{1}{4\sqrt{2}}x - \frac{9}{4\sqrt{2}}.$$

5. **Extra credit problem.** In this problem, you will use the Product Rule to prove the Power Rule.

   (a) (4 pts) Use the definition of the derivative to show that

   $$\frac{d}{dx}x = 1.$$

   $$\frac{d}{dx}x = \lim_{h\to 0}\frac{(x+h)-x}{h} = \lim_{h\to 0}\frac{h}{h} = \lim_{h\to 0} 1 = 1$$

   (b) (6 pts) Use the Product Rule to find $\frac{d}{dx}x^2$, $\frac{d}{dx}x^3$, $\frac{d}{dx}x^4$, etc until you see a pattern. Use the observed pattern to figure out $\frac{d}{dx}x^n$ for any positive integer n.

   $$\begin{aligned}\frac{d}{dx}x^2 &= \frac{d}{dx}(x\cdot x) = \frac{d}{dx}x\ x + x\frac{d}{dx}x = x + x = 2x\\ \frac{d}{dx}x^3 &= \frac{d}{dx}(x^2\cdot x) = \frac{d}{dx}x^2\ x + x^2\frac{d}{dx}x = 2xx + x^2 = 2x^2 + x^2 = 3x^2\\ \frac{d}{dx}x^4 &= \frac{d}{dx}(x^3\cdot x) = \frac{d}{dx}x^3\ x + x^3\frac{d}{dx}x = 3x^2x + x^3 = 3x^3 + x^3 = 4x^3\end{aligned}$$

   The pattern that appears is $\frac{d}{dx}x^n = nx^{n-1}$. In fact, we can see why the pattern must continue. If it is in fact true that

   $$\frac{d}{dx}x^n = nx^{n-1}$$

   for some positive integer n, then

   $$\begin{aligned}\frac{d}{dx}x^{n+1} &= \frac{d}{dx}(x^n\cdot x)\\ &= \frac{d}{dx}x^n\ x + x^n\frac{d}{dx}x\\ &= nx^{n-1}x + x^n\\ &= nx^n + x^n\\ &= (n+1)x^n.\end{aligned}$$

   That is the pattern continues to hold for $n+1$. And then for $n+2$, $n+3$, etc. Since we saw that $\frac{d}{dx}x^n = nx^{n-1}$ holds when $n = 1$ and $n = 2$, it must also be true for $n = 3, 4, 5, \ldots$, and ultimately for any positive integer.

   What we just did is a perfectly rigorous method of proving $\frac{d}{dx}x^n = nx^{n-1}$ for any positive integer n. It is called mathematical induction.