# MCS 119 Final Exam Solutions

May 27, 2019

1. (10 pts) A function $f$ is a ratio of quadratic functions and has a vertical asymptote $x = 4$ and just one $x$-intercept, $x = 1$. It is known that $f$ has a removable discontinuity at $x = -1$ and $\lim_{x\to -1} f(x) = 2$. Evaluate $f(0)$ and $\lim_{x\to\infty} f(x)$.

   So $f(x) = p(x)/q(x)$ where $p$ and $q$ are both quadratic polynomials. That $f$ has a vertical asymptote at 4 tells us that its value approaches $\infty$ or $-\infty$ as $x \to 4^+$ or $x \to 4^-$. The reason $p(x)/q(x)$ would go toward $\infty$ or $-\infty$ is that $q(x) \to 0$ while $p(x)$ approaches some nonzero number as $x$ gets close to 4. We know polynomials are continuous functions, so as $x$ approaches 4 (from the left or the right or both), $q(x)$ must approach $q(4)$. Hence $q(4) = 0$. So $x - 4$ is a factor of $q$.

   That $x = 1$ is an $x$-intercept tells us that $f(1) = 0$. Hence $p(1) = 0$ and $q(1) \neq 0$. So $x - 1$ must be a factor of $p$.

   That $f$ has a removable discontinuity at $x = -1$, tells us that $f$ does not have a value at $x = -1$, and hence $q(-1)$ must be 0. But the discontinuity is removable, which means that $\lim_{x\to -1} f(x)$ exists. This is only possible if $p(x)$ also approaches 0 as $x \to -1$. By the continuity argument above, $p(-1) = 0$. Hence $x + 1$ is a factor of both $p(x)$ and $q(x)$. We now know that

   $$p(x) = a(x-1)(x+1) \text{ and } q(x) = b(x-4)(x+1)$$

   for some nonzero numbers $a$ and $b$.

   So

   $$\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{p(x)}{q(x)} = \lim_{x\to -1} \frac{a(x-1)(x+1)}{b(x-4)(x+1)}.$$

   We can cancel $x + 1$ as $x \to -1$ means $x$ is close to -1 but $x \neq -1$, and hence $x + 1 \neq 0$.

   $$\lim_{x\to -1} \frac{a(x-1)}{b(x-4)} = \frac{a(-2)}{b(-5)} = \frac{2}{5}\,\frac{a}{b}$$

   This must be 2, so $a/b = 5$. Hence $f(x) = 5\frac{(x+1)(x-1)}{(x+1)(x-4)}$. Therefore

   $$f(0) = 5\frac{-1}{-4} = \frac{5}{4}.$$

   For the limit at $\infty$,

   $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left[5\,\frac{(x+1)(x-1)}{(x+1)(x-4)}\right] = \lim_{x\to\infty}\left[5\,\frac{x-1}{x-4}\right] = \lim_{x\to\infty}\left[5\,\frac{1-\frac{1}{x}}{1-\frac{4}{x}}\right] = 5$$

   because as $x \to \infty$, both $1/x \to 0$ and $4/x \to 0$.

2. (5 pts each) Let $g(x) = \sqrt[3]{x}^2 = x^{2/3}$.
   (a) Use the definition of the derivative to find $g'(a)$ if $a \neq 0$. (Hint: the algebraic identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ may be useful.)

   By the definition of the derivative,

   $$g'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{x^{2/3} - a^{2/3}}{x - a}.$$

   Let's work on $\frac{x^{2/3}-a^{2/3}}{x-a}$. Notice that

   $$x^{2/3} - a^{2/3} = \sqrt[3]{x}^2 - \sqrt[3]{a}^2 = \left(\sqrt[3]{x} - \sqrt[3]{a}\right)\left(\sqrt[3]{x} + \sqrt[3]{a}\right)$$

and

$$x-a=\sqrt[3]{x}^3-\sqrt[3]{a}^3=\left(\sqrt[3]{x}-\sqrt[3]{a}\right)\left(\sqrt[3]{x}^2+\sqrt[3]{x}\sqrt[3]{a}+\sqrt[3]{a}^2\right)$$

So

$$\frac{x^{2/3}-a^{2/3}}{x-a}=\frac{\cancel{\left(\sqrt[3]{x}-\sqrt[3]{a}\right)}\left(\sqrt[3]{x}+\sqrt[3]{a}\right)}{\cancel{\left(\sqrt[3]{x}-\sqrt[3]{a}\right)}\left(\sqrt[3]{x}^2+\sqrt[3]{x}\sqrt[3]{a}+\sqrt[3]{a}^2\right)}=\frac{\sqrt[3]{x}+\sqrt[3]{a}}{\sqrt[3]{x}^2+\sqrt[3]{x}\sqrt[3]{a}+\sqrt[3]{a}^2}$$

as long as $\sqrt[3]{x}-\sqrt[3]{a}\neq 0$. But that would only happen if $x=a$. And as $x\to a$, $x$ gets arbitrarily close to $a$, but is never equal to $a$. So

$$g'(a)=\lim_{x\to a}\frac{\sqrt[3]{x}+\sqrt[3]{a}}{\sqrt[3]{x}^2+\sqrt[3]{x}\sqrt[3]{a}+\sqrt[3]{a}^2}.$$

We can evaluate this limit by direct substitution, since it is a quotient of sums of products of continuous functions, such as cube roots and constant functions and hence is continuous by Theorems 5 and 6 in Section 1.5 as long as the denominator is not 0. Therefore

$$g'(a)=\frac{\sqrt[3]{a}+\sqrt[3]{a}}{\sqrt[3]{a}^2+\sqrt[3]{a}\sqrt[3]{a}+\sqrt[3]{a}^2}=\frac{2\sqrt[3]{a}}{3\sqrt[3]{a}^2}=\frac{2}{3}\frac{1}{\sqrt[3]{a}}=\frac{2}{3}a^{-1/3}$$

when $\sqrt[3]{a}\neq 0$, which is true if $a\neq 0$, and $a\neq 0$ was given.

(b) Show that $g'(0)$ does not exist. (Hint: use the definition of the derivative to express $g'(0)$ and show that the limit does not exist.)

For this, we need to look at

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^{2/3}-0}{x-0}=\lim_{x\to 0}\frac{x^{2/3}}{x}=\lim_{x\to 0}\frac{1}{\sqrt[3]{x}}.$$

As $x$ approaches 0 from the right, $\sqrt[3]{x}$ gets closer and closer to 0 while remaining a positive number, so $1/\sqrt[3]{x}$ becomes larger and larger, going toward $\infty$. Similarly, as $x$ approaches 0 from the left, $\sqrt[3]{x}$ gets closer and closer to 0 while remaining a negative number, so $1/\sqrt[3]{x}$ becomes more and more negative, going toward $-\infty$. So neither the left nor the right limit exists, and hence the two-sided limit cannot exist either.

Note that we can interpret this is terms of what happens to the secant line between 0 and $x$ as $x$ approaches 0. The slope of the secant line is

$$\frac{f(x)-f(0)}{x-0}=\frac{x^{2/3}-0}{x-0}=\frac{x^{2/3}}{x}=\frac{1}{\sqrt[3]{x}}.$$

As $x\to 0^+$, $1/\sqrt[3]{x}$ becomes larger and larger, indicating that the tangent line is becoming very steep and in the limit vertical.
Similarly, as $x\to 0^-$, $1/\sqrt[3]{x}$ becomes more and more negative, indicating that the tangent line is becoming very steep and in the limit vertical. So the tangent line would have to be vertical at $x=0$ and the slope of a vertical line is not defined.

Also note that you cannot just say that if we try to substitute $a=0$ into $f'(a)=\frac{2}{3}\frac{1}{\sqrt[3]{a}}$ then we get 0 in the denominator, hence $f'(0)$ does not exist. The formula $f'(a)=\frac{2}{3}\frac{1}{\sqrt[3]{a}}$ that we got in part (a) is based on the existence of

$$\lim_{x\to a}\frac{x^{2/3}-a^{2/3}}{x-a}.$$

In evaluating this limit, we used the direct substitution $x=a$ and noted that this is legitimate as long as the denominator does not become 0. The point of part (b) is exactly to show that this limit does not exist if $a = 0$. And if it does not exist, then it cannot be equal to $\frac{2}{3}\frac{1}{\sqrt[3]{a}}$.

3. (10 pts) If

$$\sqrt{x} + \sqrt{y} = 1$$

use implicit differentiation to find $y''$.

$$\begin{aligned}
\frac{d}{dx}\left(\sqrt{x} + \sqrt{y}\right) &= \frac{d}{dx}1 \\
\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\,y' &= 0 \\
\frac{1}{2\sqrt{y}}\,y' &= -\frac{1}{2\sqrt{x}} \\
y' &= -\frac{2\sqrt{y}}{2\sqrt{x}} \\
&= -\frac{\sqrt{y}}{\sqrt{x}}
\end{aligned}$$

So

$$y'' = \frac{d}{dx}y' = \frac{d}{dx}\left[-\frac{\sqrt{y}}{\sqrt{x}}\right] = -\frac{\frac{1}{2\sqrt{y}}\,y'\sqrt{x} - \left(\sqrt{y}\,\frac{1}{2\sqrt{x}}\right)}{\sqrt{x}^2} = -\frac{\frac{\sqrt{x}}{\sqrt{y}}\,y' - \frac{\sqrt{y}}{\sqrt{x}}}{2x}$$

by the quotient rule. We can now substitute $y' = -\frac{\sqrt{y}}{\sqrt{x}}$ and simplify the expression:

$$y'' = -\frac{\frac{\sqrt{x}}{\sqrt{y}}\left(-\frac{\sqrt{y}}{\sqrt{x}}\right) - \frac{\sqrt{y}}{\sqrt{x}}}{2x} = \frac{1 + \frac{\sqrt{y}}{\sqrt{x}}}{2x} = \frac{\frac{\sqrt{x}}{\sqrt{x}} + \frac{\sqrt{y}}{\sqrt{x}}}{2x} = \frac{\sqrt{x} + \sqrt{y}}{2x\sqrt{x}}.$$

And since the original equation of this curve was $\sqrt{x} + \sqrt{y} = 1$, we have

$$y'' = \frac{1}{2x\sqrt{x}}.$$

4. (10 pts) Use the product rule and the chain rule to prove the quotient rule.

Here is how we did this in class:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{d}{dx}\left(f(x)[g(x)]^{-1}\right).$$

By the product rule,

$$\frac{d}{dx}\left(f(x)[g(x)]^{-1}\right) = f'(x)[g(x)]^{-1} + f(x)\frac{d}{dx}[g(x)]^{-1}.$$

By the chain rule,

$$\frac{d}{dx}[g(x)]^{-1} = (-1)[g(x)]^{-2}g'(x) = -\frac{g'(x)}{[g(x)]^2}.$$

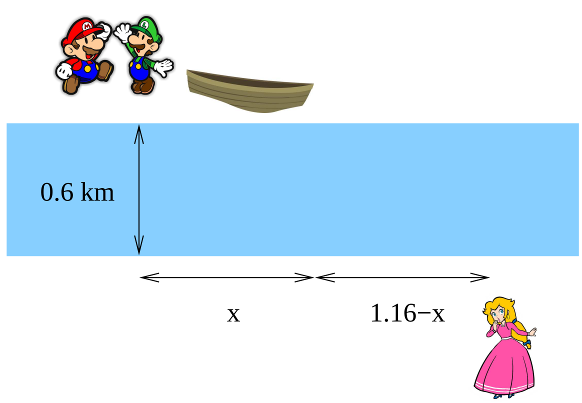
Now, let us put the pieces together:

$$\begin{aligned}\frac{d}{dx}\frac{f(x)}{g(x)} &= f'(x)[g(x)]^{-1}] + f(x)\left(-\frac{g'(x)}{[g(x)]^2}\right) \\ &= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} \\ &= \frac{f'(x)}{g(x)}\frac{g(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}\end{aligned}$$

which is the usual form of the quotient rule.

5. (10 pts) Mario and Luigi are on the bank of a crocodile-infested canal. The canal is 600 m wide. On the opposite bank of the canal, 1160 m from the point directly across from Mario's and Luigi's position is a damsel in distress for them to rescue. Mario and Luigi have a boat that they can row at a speed of 5 km/h. They can row the boat directly across the river or they can row to some point that is closer to the damsel. Once they reach the opposite bank, Mario and Luigi can run at a speed of 13 km/h. What is fastest that Mario and Luigi can get to the damsel?

Mario and Luigi are technically savvy plumbers, so they know an optimization problem when they see one and are quick to apply the calculus they learned in plumber school. Since the speeds are given in km/h, we will use km as units when we talk about distances. Suppose Mario and Luigi row their boat to a place that is between the point directly across from where they start and the damsel. Let this place be $x$ km from the point directly across from them:




By the Pythagorean Theorem, they would have to row $\sqrt{0.6^2 + x^2}$ km. Then they would have to run $1.16 - x$ km. This will take them

$$f(x) = \frac{\sqrt{0.6^2 + x^2}}{5} + \frac{1.16 - x}{13}$$

hours. We need to find the absolute minimum of $f(x)$. Notice that only values $0 \le x \le 1.16$ make sense as rowing away from the damsel or past the damsel will definitely not make the journey fastest. So we are to find the absolute minimum of $f(x)$ over the closed interval $[0, 1.16]$. As we noted many times, such an absolute minimum must be either at a critical point of $f$ inside the interval or one of the endpoints of the interval. Now

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(\frac{1}{5}\sqrt{0.6^2+x^2}+\frac{1}{13}(1.16-x)\right)\\
&= \frac{1}{5}\frac{d}{dx}(0.6^2+x^2)^{1/2}+\frac{1}{13}\frac{d}{dx}(1.16-x)\\
&= \frac{1}{5}\frac{1}{2}(0.6^2+x^2)^{-1/2}\frac{d}{dx}(0.6^2+x^2)+\frac{1}{13}(-1)\\
&= \frac{1}{5}\frac{1}{2}(0.6^2+x^2)^{-1/2}2x-\frac{1}{13}\\
&= \frac{1}{5}\frac{x}{\sqrt{0.6^2+x^2}}-\frac{1}{13}
\end{aligned}$$

Note that $0.6^2+x^2$ is always going to be positive. So there is never a negative number under the square root and never a 0 in the denominator. Hence $f'(x)$ exists for any value of $x$. Therefore the only kind of critical point is where $f'(x)=0$:

$$\begin{aligned}
\frac{1}{5}\frac{x}{\sqrt{0.6^2+x^2}}-\frac{1}{13} &= 0\\
\frac{1}{5}\frac{x}{\sqrt{0.6^2+x^2}} &= \frac{1}{13}\\
13\frac{x}{\sqrt{0.6^2+x^2}} &= 5\\
13x &= 5\sqrt{0.6^2+x^2}
\end{aligned}$$

where we could multiply both sides by $\sqrt{0.6^2+x^2}$ since we know $\sqrt{0.6^2+x^2}$ cannot be 0.

$$\begin{aligned}
13x &= 5\sqrt{0.6^2+x^2}\\
(13x)^2 &= \left(5\sqrt{0.6^2+x^2}\right)^2\\
169x^2 &= 25(0.6^2+x^2)\\
169x^2 &= 25(0.6^2+x^2)\\
169x^2 &= 9+25x^2\\
144x^2 &= 9\\
x^2 &= \frac{9}{144}=\frac{1}{16}\\
x &= \pm\frac{1}{4}
\end{aligned}$$

Now, $x=-1/4$ is outside the interval $[0,1.16]$. So the only critical point that concerns us is $x=1/4$. Let us check the values of $f$ at this critical point and at the two endpoints:

$$\begin{aligned}
f(0) &= \frac{\sqrt{0.6^2}}{5}+\frac{1.16}{13}\approx 0.12+0.089=0.209\\
f(1.16) &= \frac{\sqrt{0.6^2+1.16^2}}{5}+\frac{1.16-1.16}{13}\approx 0.261+0=0.261\\
f(0.25) &= \frac{\sqrt{0.6^2+0.25^2}}{5}+\frac{1.16-0.25}{13}=0.13+0.07=.2
\end{aligned}$$

The smallest of these is 0.2, so the fastest Mario and Luigi can get to the damsel is 0.2 hours or 12 minutes.

6. (5 pts each)

(a) Evaluate the integral

$$\int_{10}^{1} x^3 - \frac{3}{x^2} + \sin(\pi x)\, dx.$$

First, we will find the antiderivative of $x^3 - \frac{3}{x^2} + \sin(\pi x)$. We know

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \qquad \text{if } n \neq -1$$

$$\int \sin(x)\, dx = -\cos(x) + c$$

Hence

$$\int x^3 \, dx = \frac{x^4}{4} + c$$

$$\int \frac{3}{x^2}\, dx = \int 3x^{-2}\, dx = 3\frac{x^{-2+1}}{-2+1} + c = -\frac{3}{x} + c$$

As for $\sin(\pi x)$, we will try differentiating $-\cos(\pi x)$:

$$\frac{d}{dx} - \cos(\pi x) = -(-\sin(\pi x))\frac{d}{dx}(\pi x) = \sin(\pi x)\pi$$

because of the chain rule. So we are off by a factor of $\pi$ from what we want. Hence we will divide by $\pi$ and indeed,

$$\frac{d}{dx} - \frac{\cos(\pi x)}{\pi} = \frac{1}{\pi}\sin(\pi x)\pi = \sin(\pi x).$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned}
\int_{10}^{1} x^3 - \frac{3}{x^2} + \sin(\pi x)\, dx &= \frac{x^4}{4} - \frac{3}{x} - \frac{\cos(\pi x)}{\pi}\Big|_{10}^{1} \\
&= \frac{1^4}{4} - \frac{3}{1} - \frac{\cos(\pi 1)}{\pi} - \left(\frac{10^4}{4} - \frac{3}{10} - \frac{\cos(\pi 10)}{\pi}\right) \\
&= \frac{1}{4} - 3 - \frac{-1}{\pi} - 2500 + \frac{3}{10} + \frac{1}{\pi} \\
&= 2497.55 + \frac{2}{\pi}
\end{aligned}$$

(b) Let $f(x) = \ln(x^2 + 8)$. Find all values of $x$ where the tangent line to $f$ has slope $-1/3$.

We need to find the values of $x$ for which $f'(x) = -1/3$.

$$f'(x) = \frac{1}{x^2+8}\frac{d}{dx}(x^2+8) = \frac{1}{x^2+8}2x = \frac{2x}{x^2+8}.$$

This is $-1/3$ when

$$\begin{aligned}
\frac{2x}{x^2+8} &= -\frac{1}{3} \\
-6x &= x^2 + 8 \\
x^2 + 6x + 8 &= 0 \\
(x+2)(x+4) &= 0
\end{aligned}$$

So the values of $x$ we are after are $x = -2$ and $x = -4$.

7. (5 pts each) **Extra credit problem.**

(a) Let

$$F(x) = \int_1^{x^2+8} \frac{1}{t}\, dt.$$

Use a left-hand sum and a right-hand sum with seven rectangles of equal width to give upper and lower estimates for the value of $F(0)$.

$$F(0) = \int_1^{0^2+8} \frac{1}{t}\, dt = \int_1^{8} \frac{1}{t}\, dt$$

If we divide up the interval $[1, 8]$ into seven equal pieces, then each has width 1. So the left-hand sum is

$$L = (f(1) + f(2) + \cdots + f(7)) \cdot 1 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \frac{363}{140} \approx 2.593$$

and the right-hand sum is

$$R = (f(2) + f(3) + \cdots + f(8)) \cdot 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{481}{280} \approx 1.719$$

Since $1/t$ is a decreasing function on $(0, \infty)$, the left-hand sum is an upper estimate and the right-hand sum is a lower estimate for the area under the graph. Note that the two results are quite far apart, which is not surprising since using rectangles of width as large as 1 is unlikely to give a precise estimate.

(b) On what interval(s) of $\mathbb{R}$ is $F$ increasing? On what interval(s) is it decreasing?

We can tell this by looking at the derivative of $F$. Let

$$G(x) = \int_1^{x} \frac{1}{t}\, dt$$

Then $G'(x) = 1/x$ by the Fundamental Theorem of Calculus. Now, $F(x) = G(x^2 + 8)$. So by the chain rule,

$$F'(x) = G'(x^2 + 8)\, \frac{d}{dx}(x^2 + 8) = \frac{1}{x^2 + 8}\, 2x = \frac{2x}{x^2 + 8}.$$

Wherever this is positive, $F$ is increasing; wherever it is negative, $F'$ is decreasing. Note that $x^2 + 8 > 0$ for all real numbers $x$, so the sign of $\frac{2x}{x^2+8}$ is determined by the sign of the numerator. That is $F'(x) > 0$ whenever $2x > 0$, which is whenever $x > 0$; and $F'(x) < 0$ whenever $2x < 0$, which is whenever $x < 0$. So $F$ is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

Look at that, $F'(x)$ is the same as the derivative of $\ln(x^2 + 8)$ was in problem 6.b! This is not surprising at all. In fact, we know that

$$G(x) = \int_1^{x} \frac{1}{t}\, dt = \ln(x),$$

so $F(x) = G(x^2 + 8) = \ln(x^2 + 8)$. In fact, we could have used this fact to differentiate $F$ exactly as we did in problem 6.b.

(c) Find the second derivative of $F$ and use it to identify the inflection points of $F$.

We can find the inflection points by looking for those values of $x$ where $F''(x)$ changes its sign.

$$\begin{aligned} F''(x) &= \frac{d}{dx}\frac{2x}{x^2+8} \\ &= \frac{2(x^2+8)-2x(2x)}{(x^2+8)^2} \\ &= \frac{2x^2+16-4x^2}{(x^2+8)^2} \\ &= \frac{16-2x^2}{(x^2+8)^2} \end{aligned}$$

Note that the denominator of $F''(x)$ is always positive. So $F''(x)$ changes its sign whenever it numerator does. The numerator is $16-2x^2$, whose graph is an upside down parabola. It changes its sign wherever it crosses the $x$-axis, which is at

$$0 = 16 - 2x^2 \implies 2x^2 = 16 \implies x^2 = 8 \implies x = \pm\sqrt{8}.$$

So $F$ has inflection points at $x = \pm\sqrt{8} = \pm 2\sqrt{2}$.