MCS 119 Review Sheet

Here is a list of topics we have covered so far. Your review should certainly include your homework problems, as some of these will show up on the exam. You should also review the problems in your Webwork homework as the problem solving strategies you practiced while working on those may prove useful in solving problems on the exam.

The word "understand" is often used below. The definition of understanding is that you understand something when you know why it is true and can give a coherent and correct argument (proof) to convince someone else that it is true.

- Functions
  - Definition of a function, what makes a rule a function, examples and non-examples. Four standard ways to represent a function: verbal description, formula, table of values, graph.
  - Domain, codomain, and range.
  - Functions whose domain and codomain are subsets of $\mathbb{R}$.
    - The vertical line test.
    - Piecewise defined functions, examples, the absolute value function.
    - Even and odd functions, definitions, examples, symmetries of their graphs.
    - Increasing and decreasing functions, definitions, examples.
- A catalog of essential functions.
  - Linear functions, slope, y-intercept, how to find the equation of a line.
  - Power functions $f(x) = x^n$, how to make sense of positive and negative integer exponents, rational number exponents and roots, irrational exponents.
  - Families of curves as the exponent changes in $f(x) = x^n$, shape of the graph if $n > 1$, if $0 < n < 1$, if $n < 0$.
  - Polynomials, coefficients, degree, leading term, graphs of polynomials.
  - Rational functions, domain of a rational function, graphs of rational functions, vertical asymptotes and other discontinuities.
  - Exponential functions $f(x) = a^x$, shape of the graph for different values of the base, $a > 0$.
  - Logarithmic functions, definition of $\log_a(x)$, connection to exponential functions, shape of the graph depending on a.
  - Trigonometric functions, definitions in a right triangle, definitions extended via the unit circle, graphs of trig functions and their symmetries, periodic behavior, a few fundamental trig identities, such as $\cos(x) = \sin(\pi/2 - x)$, $\tan(x) = \sin(x)/\cos(x)$, and $\sin^2(x) + \cos^2(x) = 1$.
- New functions from old functions
  - Transformations of functions: $g(x) = f(x) + c$, $g(x) = cf(x)$, $g(x) = f(x + c)$, and $g(x) = f(cx)$. Understand what these do to the graph of f depending on the value of c. Also, understand how you can combine these transformation (i.e. in what order to do the shift, stretch/compression, reflection, if the order matters).
  - Composition of functions, the meaning of $f \circ g$, when can you compose two functions and when can you not, how composition affects the domain.
- Limits of functions
  - The informal meaning of $\lim_{x\to a} f(x) = L$. Understand what it means that $f(x)$ gets arbitrarily close to L as x gets sufficiently close to a. Understand what it means when

the limit does not exist. Can you think of some examples of functions whose limits do not exist somewhere?

  - Guessing limits by substituting values of x near a into $f(x)$ and the dangers of this approach (remember $\lim_{x\to 0}\sin(\pi/x)$). Using the graph of f to guess the limit and the dangers of this approach.
  - One-sided limits, informal definitions of left and right limits. Understand what the definitions mean, how they are similar to the ordinary limit, and how they are different.
  - The connection between ordinary limit and one-sided limits: $\lim_{x\to a} f(x)$ exists and equals a number L if and only if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist and equal L. Do you know how to prove this?
  - The formal definition of the limit. Understand what the $\delta - \epsilon$ definition means and how proving $\lim_{x\to a} f(x) = L$ using a $\delta - \epsilon$ argument is different from guessing the limit. Be able to construct $\delta - \epsilon$ arguments to prove $\lim_{x\to a} f(x) = L$ in some simple cases, such as for a linear function of the form $f(x) = 3x - 5$.
  - Understand how a function f that does not have a limit at a point a would fail to satisfy the $\delta - \epsilon$ definition. Can you think of an example?
  - The Limit Laws. Understand why they hold, how to use them, and when they cannot be used.
  - The Direct Substitution Property for limits of polynomials and rational functions. Understand why it holds (proof) when it can be used and when it cannot.
  - Replacing $f(x)$ with another function $g(x)$ in $\lim_{x\to a} f(x)$ as long as $f(x) = g(x)$ for all x except possibly at $x = a$. Understand why this is allowed and how it can be used to evaluate limits.
  - Theorem: If $f(x) \le g(x)$ for every x in some neighborhood of a and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$. Understand why this is true (proof).
  - The Squeeze Theorem. Understand why it holds (proof) and how it can be used to evaluate limits.
  - The limit of a composite function: if $\lim_{x\to a} g(x) = b$ and $\lim_{x\to b} f(x) = L$, then $\lim_{x\to a} f(g(x)) = L$.
  - Using tools strategically to evaluate limits. Understand how the Limit Laws, the Direct Substitution Property, and other results about limits can be combined with appropriate algebraic manipulations (e.g. factoring, reducing and expanding fractions, etc) to evaluate limits.

- Continuity
  - Definition f being continuous at a in terms of the limit. Understand how this is the same as the Direct Substution Property. Examples of different kinds of discontinuities.
  - Continuity over an interval.
  - Left and right continuity, definitions, examples.
  - Adding and subtracting continuous functions, multiplying by a scalar, multiplying, and dividing continuous functions.
  - Polynomials, rational functions, root functions, and trigonometric functions are continuous at any number x in their domains.
  - If g is continuous at a, then $\lim_{x\to a} f(g(x)) = f\big(\lim_{x\to a} g(x)\big)$. The composition of two continuous functions is continuous.
  - The Intermediate Value Theorem, examples and applications.

- Infinite limits and limits at infinity
  - Informal definitions of
    * $\lim_{x\to a} f(x) = \infty$

    * $\lim_{x\to a} f(x) = -\infty$
    * $\lim_{x\to \infty} f(x) = L$
    * $\lim_{x\to -\infty} f(x) = L$
    * $\lim_{x\to \infty} f(x) = \infty$
    * $\lim_{x\to -\infty} f(x) = \infty$
    * $\lim_{x\to \infty} f(x) = -\infty$
    * $\lim_{x\to -\infty} f(x) = -\infty$

  and corresponding one-sided limits. Notice there are many of these, so rather than memorizing definitions, find a good system to be able construct the definition for yourself. Understanding why the definition says what it says is key.
  - Intuitive understanding of infinite limits based on what the values of a function do and what the graph looks like
  - A few computational tricks, such as $\lim_{x\to\infty} \frac{p(x)}{q(x)}$ and $\lim_{x\to\infty} \frac{p(x)}{q(x)}$ when p and q are polynomials.
  - Vertical and horizontal asymptotes, definitions and how to find them
  - Formal definitions of infinite limits with M, N, δ, ϵ. Again, there are are many combinations, especially if you include one-sided limits, so find a good system to be able to reconstruct these in your mind instead of memorizing. Proving what the limit is using the formal definition.

- Rates of change
  - The slope of a secant line and average rate of change. What it means, proper units, and how to calculate it using the difference quotient. Equation of the secant line.
  - The slope of the tangent line and instantaneous rate of change. The tangent line at $x = x_1$ is the limit of the secant lines between x_1 and x_2 as $x_2 \to x_1$. Physical units. Equation of the tangent line.
  - Definition of the derivative:
    * $f'(a) = \lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$
    * $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

    Using the above definitions to find the slope of f. The limit can often be evaluated by the computational tools we have for working with limits: factoring, rationalizing roots, direct substitution into continuous functions, and the limit laws.
  - Various notations for the derivative: f', $f'(x)$, $\frac{df}{dx}$, $\frac{dy}{dx}$, $\lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x}$.
  - The derivative as a function. The connection between the graph of f and the graph of f'. Understand why $f > 0$ if f is increasing, $f' < 0$ if f is decreasing, and $f' = 0$ if the tangent line is horizontal.
  - Differentiability of f. If f is differentiable at x then f must be continuous at x. Proof.
  - The reasons why $f'(x)$ may not exist. Some of the most typical reasons are: f is discontinuous, f has a vertical tangent line, the graph of f has a sharp corner (cusp). Understand why these make the limit of the difference quotient not exist. Note that these are not the only reasons that $f'(x)$ may not exist.
  - Higher derivatives. Notation $f''(x) = \dfrac{d^2 f}{dx^2}$, $f^{(n)}(x) = \dfrac{d^n f}{dx^n}$, etc.
  - f'' as the rate of change of f'. Roughly speaking, f'' is the acceleration of f. Understand why
    * $f'' > 0$ if f' is increasing, which happens if the graph of f is curving up,
    * $f'' < 0$ if f' is decreasing, which happens if the graph of f is curving down.
  - Basic differentiation formulas:

    * Constant function: $\frac{d}{dx}c = 0$. Intuition: f is constant, its value does not change, so its rate of change is 0. Proof using the definition of the derivative.
    * Power function: $\frac{d}{dx}x^n = nx^{n-1}$. Holds for any $n \in \mathbb{R}$. Proof for $n \in \mathbb{Z}^+$ using the definition of the derivative.
    * Constant multiple: $\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$ if f is differentiable at x. Interpretation in terms of vertically stretching/compressing the graph of f. Proof using the definition of the derivative.
    * Sum of functions: $\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx} + dgdx$ if f and g are differentiable at x. Proof using the definition of the derivative.
    * Difference of functions: $\frac{d}{dx}[f(x) - g(x)] = \frac{df}{dx} - dgdx$ if f and g are differentiable at x. Proof using the definition of the derivative.
    * The product rule: $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$ if f and g are differentiable at x. Proof using the definition of the derivative.
    * The quotient rule: $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$ if f and g are differentiable at x. Proof using the product rule and the chain rule.
    * The chain rule: $\frac{d}{dx}f \circ g(x) = f'\big(g(x)\big)\, g'(x)$ if f is differentiable at $g(x)$ and g is differentiable at x. Somewhat defective proof by using the definition of the derivative and multiplying by $\frac{g(x) - g(a)}{g(x) - g(a)}$ or by $\frac{g(x+h) - g(x)}{g(x+h) - g(x)}$. Understand why this proof is not quite correct.
    * The derivatives of trigonometric functions:

      $$\frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\cos(x) = -\sin(x) \qquad \frac{d}{dx}\tan(x) = \sec^2(x)$$

      $$\frac{d}{dx}\sec(x) = \sec(x)\tan(x) \qquad \frac{d}{dx}\csc(x) = -\csc(x)\cot(x) \qquad \frac{d}{dx}\cot(x) = -\csc^2(x)$$

      Proof of $\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}\cos(x) = -\sin(x)$ using approrpiate trig identities and the unit circle. Proofs of the other four by using the quotient rule.
  - Implicit differentiation. Understand how you can find the slope of a curve given by an equation in x and y by differentiating with respect to x while treating $y = y(x)$ as an implicit function of x in a neighborhood of a point on the curve. Know how to find the equation of the tangent line and higher derivatives. Understand why a curve may not have a slope at a point (e.g. vertical tangent line, self-intersection).
  - Related rates. Understand how the rates of change of several quantities that are related by an equation are related to each other. Know how to use the chain rule to solve word problems about related rates.

- Applications of differentiation
  - Definitions of local and absolute extrema and critical points. Fermat's Theorem and its proof. The Extreme Value Theorem. The closed interval method for finding the absolute extrema of a continuous function over a closed interval $[a, b]$.
  - Rolle's Theorem. Proof using the Extreme Value Theorem and Fermat's Theorem.
  - The Mean Value Theorem. Proof using Rolle's Theorem. Understand why the function needs to be continuous on $[a, b]$ and differentiable on (a, b).

- Theorem that a function whose derivative on an interval is 0 must be a constant function. Proof using the MVT. Corollary that says if $f'(x) = g'(x)$ on some interval then $f(x) = g(x) + c$ on that interval and its proof.
- The connection between f' and f. If $f'(x) > 0$ on some interval then f is increasing on that interval; and if $f'(x) < 0$ on some interval then f is decreasing on that interval. Proofs of these last two. The First Derivative Test to determine if a critical point is a local min or max or neither.
- The connection between f'' and f. Definitions of concave up, concave down, inflection point. If $f''(x) > 0$ on some interval then f is concave up on that interval; and if $f'.(x) < 0$ on some interval then f is concave down on that interval. The Second Derivative Test to determine if a critical point is a local min or max.
- Antiderivatives. Definition of antiderivative. The role of the constant c in the most general form of the antiderivative. Know how to use the rules of differentiation backwards to find the antiderivative of a function. Rectilinear motion: using antiderivatives to recover velocity from acceleration and position from velocity.

• The area under the graph
  - Approximating the value of a function from its derivative. The distance problem: approximating distance traveled by summing the product of velocity by time over short time intervals.
  - Approximating the area under the graph of f by rectangles. The Riemann sum, partition of an interval, sample points. Special cases of the Riemann sum: left-hand sums, right-hand sums, the midpoint rule, lower sums, upper sums. Understand how to calculate a Riemann-sum approximation using a finite number of rectangles.
  - Area under the graph defined as the limit of the Riemann sum. Calculating such a limit directly from the definition, using properties of sums and limits. Definition of the definite integral and integrable function. Integral notation, dummy variables. Understand why areas under the x-axis count as negative.
  - Theorem: A function f that is continuous or has only a finite number of jump discontinuities on a closed interval $[a, b]$ is integrable on $[a, b]$.
  - Understand how the definite integral can be found in some special cases using area formulas for familiar shapes and symmetry instead of calculating limits.
  - Properties of the definite integral:
    * $\int_a^b c\,dx = c(b-a)$
    * $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
    * $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
    * $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
    * If $0 \le f(x)$ for all $x \in [a, b]$ then $0 \le \int_a^b f(x)\,dx$
    * If $f(x) \ge g(x)$ for all $x \in [a, b]$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$
    * If $m \le f(x) \le M$ for all $x \in [a, b]$ then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$
  - The definition of the indefinite integral of f as the most general antiderivative of f. Make sure you understand the difference between the definite and the indefinite integrals: the

former is the area under the graph (that is a number), the latter is the antiderivative (that is a function).
  - The Fundamental Theorem of Calculus establishes the connection between integration and differentiation: If f is a continuous function on the interval $[a,b]$ then
    1. $\displaystyle\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ for any $x \in [a,b]$.
    2. $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$, where F is an antiderivative (or indefinite integral) of f, that is $F'(x) = f(x)$.
  - Important example of using the FTC: we defined the funtion $F(x) = \int_1^x \frac{1}{t}\,dt$ and noted that this function is the natural log function $\ln(x)$, but we did not prove it. Hence $\frac{d}{dx}\ln(x) = 1/x$.
  - The average value of a function f over an interval. The Mean Value Theorem for Integrals and its proof.

- Inverse functions
  - Definition of the inverse of a function $f(x)$. How to find the inverse by using $y = f^{-1}(x)$ if and only if $x = f(y)$ and solving for y or by solving $f(g(x)) = x$ for $g(x)$.
  - The domain and range of f^{-1} and its graph as a mirror image of the graph of f.
  - One-to-one functions, definition, examples. The horizontal line test. A function f can only have an inverse if f is one-to-one.
  - Theorem: if f is a one-to-one function from domain A to range B then it has an inverse $f^{-1} : B \to A$. Informal argument by reversing arrows in a diagram. We noted that a function that is increasing or decreasing is one-to-one.
  - Theorem: If f is a continuous function that has an inverse f^{-1} then f^{-1} is also continuous. We did not prove this but observed that it makes sense by thinking about the graphs of f and f^{-1}, which are mirror images.
  - As an example of inverse functions, we noted that the $\ln(x)$ function is increasing and therefore one-to-one and therefore has an inverse function. You know from prior courses that the inverse of $\ln(x)$ is e^x.
  - If f is a differentiable function that has an inverse f^{-1}, which is also differentiable, then
  $$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'\big(f^{-1}(x)\big)}$$
  by using the chain rule.

- Derivatives and antiderivatives of exponential and logarithmic functions
  - We found $\frac{d}{dx}e^x = e^x$ by using the fact that e^x is the inverse of $\ln(x)$. So the antiderivative of e^x is $e^x + c$. To differentiate or antidifferentiate a^x, express it as $e^{x\ln(a)}$ and use the chain rule.
  - To differentiate log functions, use the FTC and the fact that $\ln(x) = \int_1^x \frac{1}{t}\,dt$. Other log functions can be differentiated by noting that $\log_a(x)$ is the inverse of a^x or by using the logarithmic identity $\log_a(x) = \frac{\ln(x)}{\ln(a)}$. We did not learn about the antiderivatives of log functions.