MCS 119 Exam 1 Solutions

1. (10 pts) Find the values of a and b that make the piecewise-defined function

$$f = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leq x < 3 \\ 2x - a + b & \text{if } x \leq 3 \end{cases}$$

continuous everywhere. Be sure to explain how you know that f is continuous <u>everywhere</u>.

First, notice that the first piece of this function $\frac{x^2-4}{x-2}$ can be simplified as

$$\frac{x^2-4}{x-2} = \frac{(x+2)(x-2)}{x-2} = x+2$$

as long as $x \neq 2$. But this piece is over the interval $(-\infty, 2)$, so $x \neq 2$. Now, $x+2$ is a polynomial and so it is continuous everywhere. In particular, it is continuous at every $x < 2$. The second piece $ax^2 - bx + 3$ is a polynomial, so it is continuous everywhere, and in particular at every $2 < x < 3$. The third piece $2x - a + b$ is a linear function, so also a polynomial, hence it is continuous everywhere, and in particular at every $3 < x$.

So the only places where f could be discontinuous are 2 and 3. We need to make sure f is continuous there too. For continuity at 2, we need

$$\lim_{x\to 2} f(x) = f(2) = 4a - 2b + 3.$$

Since f is piecewise defined and this is exactly where two of its pieces meet, we need to look at the one-sided limits. First,

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (ax^2 - bx + 3) = 4a - 2b + 3$$

by direct substitution, since $ax^2 - bx + 3$ is a polynomial. Now,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x+2) = 2 + 2 = 4$$

also by direct substitution, since $x+2$ is a linear function and hence a polynomial. For f to be continuous at 2, these two one-sided limits must be equal to each other and to $f(2) = 4a-2b+3$. The right limit is already equal to $f(2)$, so this only requires $4 = 4a-2b+3$.

Similarly, for continuity at 3, we need

$$\lim_{x\to 3} f(x) = f(3) = 6 - a + b.$$

First,

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (2x - a + b) = 6 - a + b$$

by direct substitution, since $2x - a + b$ is a linear function and hence a polynomial. Now,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (ax^2 - bx + 3) = 9a - 3b + 3$$

by direct substitution, since $ax^2 - bx + 3$ is a polynomial. Again, the one-sided limits must be equal to each other and to $f(3) = 6 - a + b$. The right limit is already equal to $f(3)$, so this only requires $9a - 3b + 3 = 6 - a + b$.

We can now solve the system of linear equations

$$\begin{aligned} 4a - 2b + 3 &= 4 & &\implies 4a - 2b = 1 \\ 9a - 3b + 3 &= 6 - a + b & &\implies 10a - 4b = 3 \end{aligned}$$

Subtracting twice the first equation from the second gives

$$\begin{aligned}
10a - 4b - 2(4a - 2b) &= 3 - 2 \\
10a - 4b - 8a + 4b &= 1 \\
2a &= 1 \\
a &= \frac{1}{2}
\end{aligned}$$

To find b, we substitute this into $4a - 2b = 1$ and get

$$2 - 2b = 1 \implies 1 = 2b \implies b = \frac{1}{2}.$$

So $a = b = 1/2$ make f continuous at 2 and 3. And we already noted that f is continuous everywhere else, so f is now continuous everywhere.

2. (a) (8 pts) Use the definition of the derivative to find the slope of the tangent line to the curve $y = 1/\sqrt{x}$ at the point where $x = a$.

$$\begin{aligned}
y'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x \to a} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}}}{x - a}.
\end{aligned}$$

Now, the numerator is

$$\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}} = \frac{\sqrt{a} - \sqrt{x}}{\sqrt{x}\sqrt{a}}.$$

So the expression in the limit is

$$\frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}}}{x - a} = \frac{\frac{\sqrt{a}-\sqrt{x}}{\sqrt{x}\sqrt{a}}}{x - a} = \frac{\sqrt{a} - \sqrt{x}}{\sqrt{x}\sqrt{a}} \frac{1}{x - a} = \frac{\sqrt{a} - \sqrt{x}}{\sqrt{x}\sqrt{a}(x - a)}.$$

We will multiply the numerator and the denominator by $\sqrt{a} + \sqrt{x}$. We can do this as long as $\sqrt{a} + \sqrt{x} \neq 0$. The only way for $\sqrt{a} + \sqrt{x} = 0$ is if $x = a = 0$. But 0 is not in the domain of $y(x) = 1/\sqrt{x}$, so neither x nor a can be 0. So

$$\begin{aligned}
\frac{\sqrt{a} - \sqrt{x}}{\sqrt{x}\sqrt{a}(x - a)} &= \frac{(\sqrt{a} - \sqrt{x})(\sqrt{a} + \sqrt{x})}{\sqrt{x}\sqrt{a}(x - a)(\sqrt{a} + \sqrt{x})} \\
&= \frac{\sqrt{a}^2 - \sqrt{x}^2}{\sqrt{x}\sqrt{a}(x - a)(\sqrt{a} + \sqrt{x})} \\
&= \frac{a - x}{\sqrt{x}\sqrt{a}(x - a)(\sqrt{a} + \sqrt{x})} \\
&= \frac{-(x - a)}{\sqrt{x}\sqrt{a}(x - a)(\sqrt{a} + \sqrt{x})} \\
&= \frac{-1}{\sqrt{x}\sqrt{a}(\sqrt{a} + \sqrt{x})}.
\end{aligned}$$

Note that the reason we were able to cancel $x - a$ is that we know $x \neq a$ as $x \to a$. Now, the limit of this is

$$
\begin{aligned}
&\lim_{x\to a} \frac{-1}{\sqrt{x}\sqrt{a}(\sqrt{a}+\sqrt{x})} \\
&\quad = \frac{-1}{\lim_{x\to a}\left[\sqrt{x}\sqrt{a}(\sqrt{a}+\sqrt{x})\right]} && \text{by LL5 and LL7} \\
&\quad = \frac{-1}{\left(\lim_{x\to a}\sqrt{x}\right)\left(\lim_{x\to a}\sqrt{a}\right)\left(\lim_{x\to a}(\sqrt{a}+\sqrt{x})\right)} && \text{by LL4} \\
&\quad = \frac{-1}{\sqrt{a}\sqrt{a}(\lim_{x\to a}\sqrt{a}+\lim_{x\to a}\sqrt{x})} && \text{by LL10, LL7, and LL1} \\
&\quad = \frac{-1}{a(\sqrt{a}+\sqrt{a})} && \text{by LL10 and LL7} \\
&\quad = -\frac{1}{2a\sqrt{a}}.
\end{aligned}
$$

Hence

$$y'(x) = -\frac{1}{2x\sqrt{x}}.$$

Alternately, you can evaluate

$$\lim_{x\to a} \frac{-1}{\sqrt{x}\sqrt{a}(\sqrt{a}+\sqrt{x})}$$

by direct substitution if you know that

$$f(x) = \frac{-1}{\sqrt{x}\sqrt{a}(\sqrt{a}+\sqrt{x})}$$

is a continuous function of x. To show that, first write it as

$$f(x) = \frac{-1}{\sqrt{x}\sqrt{a}(\sqrt{a}+\sqrt{x})} = \frac{-1}{a\sqrt{x}+\sqrt{a}x}.$$

Now, the $g(x) = -1$ in the numerator is a polynomial (of degree 0) and hence continuous (Theorem 5 or 6 in Section 1.5). In the denominator, $h(x) = \sqrt{a}x$ is a polynomial in x and hence is also continuous. Finally, $k(x) = \sqrt{x}$ is a root function, and hence continuous at every x in its domain, also by Theorem 6. Now,

$$f(x) = \frac{-1}{a\sqrt{x}+\sqrt{a}x} = \frac{g(x)}{\sqrt{a}k(x)+h(x)}$$

is continuous by Theorem 4 in Section 1.5, as long as the denominator is not 0. But the only way the denominator can be 0 is if $a = x = 0$, which we noted above cannot be the case. So

$$\lim_{x\to a} \frac{-1}{\sqrt{x}\sqrt{a}(\sqrt{a}+\sqrt{x})} = \frac{-1}{a\sqrt{a}+\sqrt{a}a} = -\frac{1}{2a\sqrt{a}}$$

by direct substitution.

(b) (2 pts) Find the equation of the tangent line at the point $\left(4, \frac{1}{2}\right)$.

The slope of the line is

$$y'(4) = -\frac{1}{2(4)\sqrt{4}} = -\frac{1}{16},$$

So its equation is $y = -\frac{1}{16}x + b$. Since it passes through $\left(4, \frac{1}{2}\right)$,

$$\frac{1}{2} = -\frac{1}{16}4 + b \implies b = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

Hence the equation of the tangent line is

$$y = -\frac{1}{16}x + \frac{3}{4}.$$

3. (a) (3 pts) Define what it means for a function $f(x)$ to have a vertical asymptote at $x = a$.

A function $f(x)$ has a vertical asymptote at $x = a$ if at least one of the following holds:

$$\lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a^+} f(x) = -\infty$$
$$\lim_{x\to a^-} f(x) = \infty$$
$$\lim_{x\to a^-} f(x) = -\infty.$$

(b) (7 pts) Find the vertical asymptotes of the rational function

$$f(x) = \frac{3x^2 + 5x - 2}{x^2 + 2x - 3}.$$

Be sure to justify your work.

First, note that $x^2 + 2x - 3 = (x + 3)(x - 1) = 0$ if $x = -3$ or $x = 1$. If $a \neq -3$ and $a \neq 1$ then $f(a) = \frac{3a^2+5a-2}{a^2+2a-3}$ is a real number, and

$$\lim_{x\to a} f(x) = f(a) = \frac{3a^2 + 5a - 2}{a^2 + 2a - 3}$$

by the direct substitution property of rational functions. So

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \frac{3a^2 + 5a - 2}{a^2 + 2a - 3}$$

and therefore $f(x)$ does not have a vertical asymptote at a. We will now show that $x = 1$ and $x = -3$ are vertical asymptotes.
As $x \to -3^+$, we have $3x^2 + 5x - 2 \to 3(-3)^2 + 5(-3) - 2 = 10$. At the same time, $x^2 + 2x - 3$ is getting closer and closer to 0, while it is always negative. We know it is negative because if x is close to -3 and just a little more than -3, then $x + 3 > 0$ and $x - 1 < 0$, and so $(x + 3)(x - 1) < 0$. Hence $f(x) = \frac{3x^2+5x-2}{x^2+2x-3}$ is a number close to 10 divided by a negative number close to 0. So it is negative and is getting more and more negative as $x \to -3^+$. Therefore

$$\lim_{x\to-3^+} \frac{3x^2 + 5x - 2}{x^2 + 2x - 3} = -\infty$$

Hence $f(x)$ has a vertical asymptote at -3.
As $x \to 1^+$, we have $3x^2 + 5x - 2 \to 3(1)^2 + 5(1) - 2 = 6$. At the same time, $x^2 + 2x - 3$ is getting closer and closer to 0, while it is always positive. We know it is positive because if x is close to 1 and just a little more than 1, then $x + 3 > 0$ and $x - 1 > 0$, and so $(x + 3)(x - 1) > 0$. Hence $f(x) = \frac{3x^2+5x-2}{x^2+2x-3}$ is a number close to 6 divided by a positive number close to 0. So it is large and is getting larger and larger as $x \to 1^+$. Therefore

$$\lim_{x\to1^+} \frac{3x^2 + 5x - 2}{x^2 + 2x - 3} = \infty$$

Hence $f(x)$ also has a vertical asymptote at 1.

4. (10 pts) Let f and g be functions of real numbers that are both differentiable at a. State and prove the Difference Rule for finding the derivative of $f - g$ at a.

   The Difference Rule is

$$\frac{d}{dx}[f(x) - g(x)] = \frac{df}{dx} - \frac{dg}{dx}.$$

   Here is the proof:

$$\begin{aligned}
\frac{d}{dx}[f(x) - g(x)] &= \lim_{h\to 0} \frac{(f-g)(x+h) - f + g(x)}{h} \\
&= \lim_{h\to 0} \frac{(f(x+h) - g(x+h) - (f(x) - g(x))}{h} \\
&= \lim_{h\to 0} \left[\frac{(f(x+h) - f(x)}{h} - \frac{g(x+h) - g(x)}{h} \right] \\
&= \lim_{h\to 0} \frac{(f(x+h) - f(x)}{h} - \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} && \text{by LL2} \\
&= f'(x) - g'(x)
\end{aligned}$$

5. (10 pts) **Extra credit problem.** In class, we observed that a function f that is continous at a point a need not be differentiable there. We know that the function $f(x) = \sqrt[3]{x}$ is continuous at 0 because it is a root function and 0 is in its domain. Show that f is not differentiable at 0. That is prove that the limit of the difference quotient does not exist at 0.

$$\begin{aligned}
f'(0) &= \lim_{x\to 0} \frac{\sqrt[3]{x} - \sqrt[3]{0}}{x - 0} \\
&= \lim_{x\to 0} \frac{\sqrt[3]{x}}{x} \\
&= \lim_{x\to 0} \frac{1}{x^{2/3}}
\end{aligned}$$

   This limit does not exist because as $x \to 0$, $x^{2/3} = \sqrt[3]{x^2}$ is getting closer and closer to 0, while it is always positive because $x^2 > 0$. So the fraction $1/x^{2/3}$ has a denominator that is becoming a smaller and smaller positive number, and hence the value of the fraction is getting larger and larger. So we can conclude

$$\lim_{x\to 0} \frac{1}{x^{2/3}} = \infty$$

   and this tells use that $f'(0)$ does not exist.