MCS 119 Exam 2 Solutions

May 4, 2020

1. (5 pts each) If f is a differentiable function, find an expression for the derivative of each of the following functions.

(a) $y = x^2 f(x)$

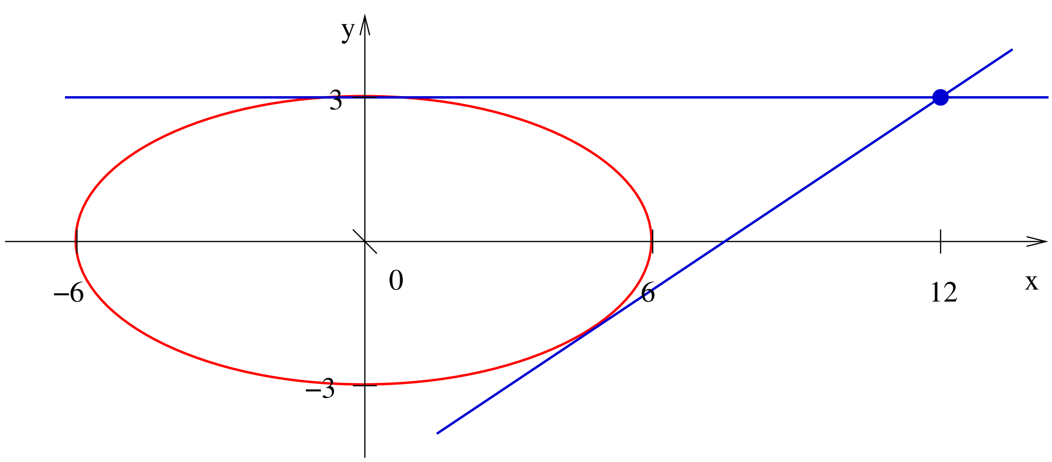
$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[x^2 f(x)] \\ &= \left(\frac{d}{dx}x^2\right) f(x) + x^2 \frac{df}{dx} && \text{by the Product Rule} \\ &= 2xf(x) + x^2 f'(x)\end{aligned}$$

(b) $y = \frac{f(x)}{x^2}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\frac{f(x)}{x^2} \\ &= \frac{f'(x)x^2 - f(x)\frac{d}{dx}x^2}{(x^2)^2} && \text{by the Quotient Rule} \\ &= \frac{x^2 f'(x) - 2xf(x)}{x^4} \\ &= \frac{xf'(x) - 2f(x)}{x^3}\end{aligned}$$

2. (10 pts) Find equations of both the tangent lines to the ellipse $x^2 + 4y^2 = 36$ that pass through the point $(12, 3)$.

First, here is a diagram:



We can find the slope of a tangent to the ellipse by implicit differentiation:

$$\frac{d}{dx}(x^2 + 4y^2) = \frac{d}{dx}36 \implies 2x + .8y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = \frac{-2x}{8y} = -\frac{x}{4y}$$

If $y = mx + b$ is a line that passes through $(12, 3)$ and a point (x_0, y_0) on the ellipse then its slope is $m = \frac{y_0 - 3}{x_0 - 12}$. If this line is also tangent to the ellipse at (x_0, y_0), then its slope is also $m = -\frac{x_0}{4y_0}$. These two must be equal, so

$$-\frac{x_0}{4y_0} = \frac{y_0 - 3}{x_0 - 12}$$

We will multiply both sides by $x_0 - 12$ and by $4y_0$. But we need to be careful because if either of those is 0, this could result in false solutions. So we will need to check in the end to make sure we did not multiply by 0.

$$\begin{aligned} -x_0(x_0 - 12) &= 4y_0(y_0 - 3) \\ -x_0^2 + 12x_0 &= 4y_0^2 - 12y_0 \\ 0 &= 4y_0^2 + x_0^2 - 12x_0 - 12y_0 \end{aligned}$$

Notice that $4y_0^2 + x_0^2 = 36$ since (x_0, y_0) is on the ellipse. Hence

$$0 = 36 - 12x_0 - 12y_0 \implies 0 = 3 - x_0 - y_0 \implies y_0 = 3 - x_0.$$

Substituting this into the equation of the ellipse gives

$$\begin{aligned} x_0^2 + 4(3 - x)^2 &= 36 \\ x_0^2 + 4(9 - 6x_0 + x_0^2) &= 36 \\ x_0^2 + 36 - 24x_0 + 4x_0^2 &= 36 \\ 5x_0^2 - 24x_0 &= 0 \\ x_0(5x_0 - 24) &= 0 \end{aligned}$$

So $x_0 = 0$ or $x_0 = 24/5$. The corresponding values of y_0 are $y_0 = 3 - 0 = 3$ and $y_0 = 3 - 24/5 = -9/5$. Note that neither $x_0 - 12 = 0$ nor $4y_0 = 0$ at either of these two points, so we were justified multiplying by $x_0 - 12$ and $4y_0$ earlier. The derivative of the ellipse at $(0, 3)$ is $y' = -0/((4)(3)) = 0$, and at $(24/5, -9/5)$, it is $y' = -\frac{24/5}{4(-9/5)} = \frac{2}{3}$. So the equations of the tangent lines are

$$\begin{aligned} y - 3 = 0(x - 0) &\implies y - 3 = 0 \implies y = 3 \\ y + \frac{9}{5} = \frac{2}{3}\left(x - \frac{24}{5}\right) &\implies y = \frac{2}{3}x - \frac{16}{5} - \frac{9}{5} = \frac{2}{3}x - 5 \end{aligned}$$

3. (10 pts) After another successful mission, James Bond is chilling at home holding a glass of martini–shaken, not stirred of course–and slowly sipping it through a straw (straw not shown in illustration). Mr. Bond's martini glass has the typical inverted cone shape with a height of 10 cm and a diameter of 10 cm at the top. As he takes a sip through the straw, Bond notes that the level of his beverage is dropping at a rate of 2 cm/min when it is 6 cm from the bottom of the glass. At what rate in cm^3/min is Mr. Bond downing his martini? Hint: recall that the volume of a cone is $V = \frac{\pi}{3}hr^2$ where r is the radius at the base and h is the height.

Let

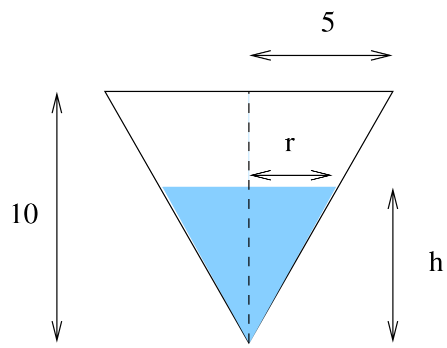
$h(t) =$ the height of Mr. Bond's drink in the glass (in cm) at time t (in min)

$V(t) =$ the volume of Mr. Bond's drink in the glass (in cm) at time t (in min)

We have the equation

$$V = \frac{\pi}{3}hr^2$$

where V and h are both functions of time. Of course, r also depends on t, but we can figure out r in terms of h using similar triangles.



So

$$\frac{r}{h} = \frac{5}{10} \implies r = \frac{h}{2}$$

at any time. Therefore

$$V = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

We can differentiate both sides with respect to t using the Chain Rule on the right-hand side:

$$\frac{dV}{dt} = \frac{\pi}{12}3h^2\frac{dh}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

We know $\frac{dh}{dt} = -2$ cm/min when $h = 6$ cm. So

$$\frac{dV}{dt} = \frac{\pi}{4}(6\text{ cm})^2\left(-2\,\frac{\text{cm}}{\text{min}}\right) = -18\pi\,\frac{\text{cm}^3}{\text{min}} \approx -56.5\,\frac{\text{cm}^3}{\text{min}}.$$

So James Bond is quaffing his martini at a rate of about 56.5 cm^3/min.

4. (10 pts) Use the Mean Value Theorem to prove that if $f'(x) = 0$ for every x in the open interval (a, b) then f is constant on (a, b).

   See Theorem 5 in Section 3.2.

5. **Extra credit problem.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is differentiable at every $x \in \mathbb{R}$.
   (a) (4 pts) Use the Chain Rule to show that if f is an odd function, then f' is an even function.

   Let f be an odd function. Then $f(-x) = -f(x)$ for every $x \in \mathbb{R}$. By differentiating both sides with respect to x, using the Chain Rule on the left-hand side, we get

$$\begin{aligned} \frac{d}{dx}f(-x) &= \frac{d}{dx}[-f(x)] \\ f'(-x)(-1) &= -f'(x) \\ f'(-x) &= f'(x) \end{aligned}$$

   So $f'(-x) = f'(x)$ for all $x \in \mathbb{R}$, and this shows f' is an even function.

   (b) (4 pts) Suppose f'' also exists at at every $x \in \mathbb{R}$. What can you say about the symmetry of f''?

Differentiating $f'(-x) = f'(x)$ again, we find

$$\begin{aligned} \frac{d}{dx} f'(-x) &= \frac{d}{dx} f'(x) \\ f''(-x)(-1) &= f''(x) \\ f''(-x) &= -f''(x) \end{aligned}$$

So $f''(-x) = -f''(x)$ for all $x \in \mathbb{R}$. Hence f'' is an odd function.

(c) (2 pts) Let n be any nonnegative integer and suppose that the n-th derivative $f^{(n)}$ exists. What symmetry must it have?

Since f'' is an odd function, its derivative f''' must be even again, then the next derivative $f^{(4)}$ will be odd, and so on, alternating between even and odd. The pattern is clear: $\frac{d^n f}{dx^n}$ is even when n is n odd and $\frac{d^n f}{dx^n}$ is odd when n is n even.