MCS 119 Final Exam Solutions

May 26, 2020

1. (10 pts) Suppose f is a function of real numbers that is continuous on $[1, 5]$ and the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) > 8$, explain why $f(3) > 6$.

   First, $f(3)$ cannot be 6 because then $x = 3$ would be another solution of $f(x) = 6$. Suppose $f(3) < 6$. Since f is continuous on $[1, 5]$ it is also continuous on $[2, 3]$. We know $f(2) > 8$ and $f(3) < 6$, therefore $f(3) < 6 < f(2)$. By the Intermediate Value Theorem, there must be some point $c \in (2, 3)$ such that $f(c) = 6$. Since $c \neq 1$ and $c \neq 4$, it would be another solution of the equation $f(x) = 6$. Assuming $f(3) < 6$ led to a contradiction, and we already noted $f(3) \neq 6$, so $f(3)$ must be greater than 6.

2. (10 pts) Find y'' by implicit differentiation for the curve

$$\sqrt{x} + \sqrt{y} = 1.$$

   By differentiating both sides of the equation, we get

$$\begin{aligned}
\frac{d}{dx}\left(\sqrt{x} + \sqrt{y}\right) &= \frac{d}{dx}1 \\
\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} &= 0 \\
\frac{1}{2\sqrt{y}}\frac{dy}{dx} &= -\frac{1}{2\sqrt{x}} \\
\frac{dy}{dx} &= -\frac{\sqrt{y}}{\sqrt{x}}
\end{aligned}$$

So

$$\begin{aligned}
y'' &= \frac{d}{dx}\frac{dy}{dx} \\
&= \frac{d}{dx}\left(-\frac{\sqrt{y}}{\sqrt{x}}\right) \\
&= -\frac{\frac{1}{2\sqrt{y}}\frac{dy}{dx}\sqrt{x} - \sqrt{y}\frac{1}{2\sqrt{x}}}{\sqrt{x}^2} \\
&= \frac{-\frac{1}{\sqrt{y}}\left(-\frac{\sqrt{y}}{\sqrt{x}}\right)\sqrt{x} + \sqrt{y}\frac{1}{\sqrt{x}}}{2x} \\
&= \frac{1 + \frac{\sqrt{y}}{\sqrt{x}}}{2x} \\
&= \frac{\frac{\sqrt{x}}{\sqrt{x}} + \frac{\sqrt{y}}{\sqrt{x}}}{2x} \\
&= \frac{\sqrt{x} + \sqrt{y}}{2x\sqrt{x}} \\
&= \frac{1}{2x\sqrt{x}} && \text{since } \sqrt{x} + \sqrt{y} = 1 \\
&= \frac{1}{2x^{3/2}}
\end{aligned}$$

3. (10 pts) Let f be a function of real numbers. Prove Fermat's Theorem for the case in which f has a local minimum at c. That is show that if f has a local minimum at c and f is differentiable at c, then $f'(c) = 0$.

Suppose f has a local minimum at $x = c$. Then $f(x) \geq f(c)$ for every x in some small enough neighborhood of c. Let us consider the right limit

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}.$$

Once x is close enough to c (and $x > c$), we have

$$f(x) \geq f(c) \implies f(x) - f(c) \geq 0 \implies \frac{f(x) - f(c)}{x - c} \geq 0$$

for every $x > c$ close to c. Hence

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \geq 0$$

if the right limit exists at all.

Similarly, let us look at the left limit

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}.$$

Once x is close enough to c (and $x < c$), we have

$$f(x) \geq f(c) \implies f(x) - f(c) \geq 0 \implies \frac{f(x) - f(c)}{x - c} \leq 0$$

for every $x < c$ close to c. Hence

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \leq 0$$

assuming the left limit exists.

Now, if f is differentiable at $x = c$, then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. But this is only possible if both the left and right limits exist and are equal. So

$$0 \leq \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \leq 0.$$

This is only possible if all four quantities equal 0. So

$$0 = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c).$$

4. (a) (8 pts) Use the definition of the derivative to find the derivative of $f(x) = 2x^3 - 5x + 4$.

$$\begin{aligned} y'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \to a} \frac{(2x^3 - 5x + 4) - (2a^3 - 5a + 4)}{x - a}. \end{aligned}$$

Now, the numerator is

$$\begin{aligned} (2x^3 - 5x + 4) - (2a^3 - 5a + 4) &= 2(x^3 - a^3) - 5(x - a) \\ &= 2(x - a)(x^2 + xa + a^2) - 5(x - a). \end{aligned}$$

So the expression in the limit is

$$\frac{2(x-a)(x^2+xa+a^2)-5(x-a)}{x-a} = 2(x^2+xa+a^2)-5$$
$$= 2x^2+2xa+2a^2-5.$$

and

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}[2x^2+2xa+2a^2-5]$$

Note that the function in the limit is a polynomial in x. Hence it is continuous everywhere and we can use direct substitution to evaluate the limit:

$$\lim_{x\to a}[2x^2+2xa+2a^2-5] = 2a^2+2aa+2a^2-5 = 6a^2-5$$

Therefore $f'(x) = 6x^2-5$.

(b) (2 pts) Find the equation of the tangent line to f at $x=3$.
The slope of the tangent line is

$$f'(3) = 6(3^2)-5 = 49.$$

So the line's equation is $y = 49x+b$. Since it passes through

$$(3, f(3)) = (3, 2(3^3)-5(3)+4) = (3, 43)$$

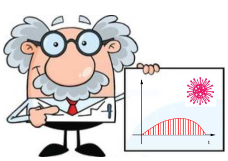
we have

$$43 = 49(3)+b \implies b = 43-147 = -104.$$

Hence the equation of the tangent line is

$$y = 49x-104.$$

5. Epidemiologists studying the spread of a novel virus have found that the function



$$f(t) = \frac{1}{1000}(-t^3+62t^2+14415t)$$

is a good model for the rate of new infections per day, where t is the time in days since the epidemic started. While the number of new cases increases initially, the model predicts that the rate of new infections should eventually slow down. The epidemic ends when the rate of new infections drops to 0. In fact, the scientists have calculated that this can be expected to happen on day 155 as $f(155) = 0$.

(a) (10 pts) On what day of the epidemic does the rate of new infections reach its peak? How do you know that what you found is indeed the peak?

We need to find the absolute maximum of the function $f(t) = \frac{1}{1000}(-t^3+62t^2+14415t)$ over the closed interval $[0, 155]$. Since f is a polynomial, it is continuous. Therefore we can use the Closed Interval Method. First,

$$f'(t) = \frac{1}{1000}(-3t^2+62(2t)+14415) = -\frac{1}{1000}(3t^2-124t-14415)$$

Note that f is a polynomial, so it is differentiable at every $t \in \mathbb{R}$. So the only kind of critical point is where $f'(t) = 0$:

$$0 = -\frac{1}{1000}(-3t^2+124t+14415) \implies 0 = 3t^2-124t-14415$$

By the quadratic formula,

$$t = \frac{124 \pm \sqrt{124^2-4(3)(-14415)}}{6} = \frac{124 \pm 434}{6} = 93, -\frac{155}{3}.$$

Of these critical points, only $t = 93$ is in $[0, 155]$. By the Closed Interval Method, we know that the absolute maximum of f is either at an interior local maximum or at one of the endpoints of $[0, 155]$. We can compare the values of f at these three points:

$$f(0) = 0$$
$$f(93) = 1072.476$$
$$f(155) = 0$$

So the peak is the absolute maximum at $t = 93$.

(b) (6 pts) Since f is the daily rate of new infections, the total number of people that have been infected by day x is

$$F(x) = \int_0^x f(t)\,dt.$$

Find the number of people that are infected during the course of the epidemic.

That number is

$$\begin{aligned} F(155) &= \int_0^{155} f(t)\,dt \\ &= \int_0^{155} \frac{1}{1000}(-t^3 + 62t^2 + 14415t)\,dt \\ &= \frac{1}{1000}\left[-\frac{t^4}{4} + 62\frac{t^3}{3} + 14415\frac{t^2}{2}\right]_0^{155} \\ &= \frac{1}{1000}\left(-\frac{155^4}{4} + 62\frac{155^3}{3} + 14415\frac{155^2}{2} - 0\right) \\ &\approx 105820 \end{aligned}$$

Therefore about 105820 people get infected during the epidemic.

(c) (4 pts) The function $F(x)$ from part (b) has an inflection point somewhere between $x = 0$ and $x = 155$. Find the inflection point and explain its significance for the epidemic.

The inflection point is where $F''(x)$ changes sign. By the Fundamental Theorem of Calculus,

$$F'(x) = \frac{d}{dx}\int_0^x f(t)\,dt = f(x).$$

Hence

$$F''(x) = f'(x) = \frac{d}{dx}\left[\frac{1}{1000}(-x^3 + 62x^2 + 14415x)\right] = -\frac{1}{1000}(3x^2 - 124x - 14415).$$

Note that this is a polynomial, so it is continuous. Therefore when it changes sign, its value is 0. That is

$$-\frac{1}{1000}(3x^2 - 124x - 14415) = 0.$$

We solved this equation in part (a) and found $x = 93$ is the only solution in $[0, 155]$. That point turned out to be the absolute maximum, and hence a local maximum of the function $f(x) = \frac{1}{1000}(-x^3 + 62x^2 + 14415x)$. Therefore the sign of $F''(x) = f'(x)$ really does change there from positive to negative at $x = 93$.

The significance of this point to the epidemic is that this is the peak of the daily infection rate. Up to this point, the daily infection rate is increasing; after this point, it is decreasing. This is the point at which the spread of the epidemic starts slowing down.

6. (10 pts) **Extra credit problem.** Let $x > 0$ and define the function $F : \mathbb{R}^+ \to \mathbb{R}$ by

$$F(x) = \int_1^x \frac{1}{t}\, dt.$$

That F has a well-defined value for every $x > 0$ follows from the fact that $f(t) = 1/t$ is a continuous function at every $t \in \mathbb{R}^+$.

(a) (8 pts) According to the Fundamental Theorem of Calculus, $F'(x) = 1/x$ for every $x \in \mathbb{R}^+$. Let $y > 0$ and define $G : \mathbb{R}^+ \to \mathbb{R}$ by

$$G(x) = \int_1^{xy} \frac{1}{t}\, dt.$$

Show that $G'(x)$ is also $1/x$ for every $x \in \mathbb{R}^+$.

$$\begin{aligned}
G'(x) &= \frac{d}{dx} \int_1^{xy} \frac{1}{t}\, dt \\
&= \frac{d}{dx} F(xy) \\
&= F'(xy)y && \text{by the Chain Rule} \\
&= \frac{1}{xy} y \\
&= \frac{1}{x}
\end{aligned}$$

(b) (5 pts) Since $G'(x) = F'(x)$, we know $G(x) = F(x) + c$ for some constant c (by Corollary 7 in Section 3.2). Prove that $c = F(y)$ by evaluating F and G at $x = 1$.

We know $G(x) = F(x) + c$. Let us substitute $x = 1$:

$$\begin{aligned}
G(1) &= F(1) + c \\
\int_1^y \frac{1}{t}\, dt &= \int_1^1 \frac{1}{t}\, dt + c \\
\int_1^y \frac{1}{t}\, dt &= 0 + c \\
c &= \int_1^y \frac{1}{t}\, dt \\
c &= F(y)
\end{aligned}$$

(c) (2 pts) Conclude that $F(xy) = F(x) + F(y)$ for any $x, y \in \mathbb{R}^+$.

Since $c = F(y)$,

$$F(xy) = \int_1^{xy} \frac{1}{t}\, dt = G(x) = F(x) + c = F(x) + F(y).$$