Formal definitions of infinite limits

Here are the formal definitions of limits that involve infinities. Your textbook does not list all of these because they are so similar to each other and because you should have enough experience with limits to be able to construct these definitions for yourself. In fact, I strongly recommend that you do exactly that and use the definitions in this handout only to check your own work.

Definition 1. Let $f(x)$ be a function. We say

$$\lim_{x \to a} f(x) = \infty$$

if for every (positive) number M there is a corresponding $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - a| < \delta$.

Definition 2. Let $f(x)$ be a function. We say

$$\lim_{x \to a} f(x) = -\infty$$

if for every (negative) number M there is a corresponding $\delta > 0$ such that $f(x) < M$ whenever $0 < |x - a| < \delta$.

Definition 3. Let $f(x)$ be a function. We say

$$\lim_{x \to a^-} f(x) = \infty$$

if for every (positive) number M there is a corresponding $\delta > 0$ such that $f(x) > M$ whenever $a - \delta < x < a$.

Definition 4. Let $f(x)$ be a function. We say

$$\lim_{x \to a^+} f(x) = \infty$$

if for every (positive) number M there is a corresponding $\delta > 0$ such that $f(x) > M$ whenever $a < x < a + \delta$.

Definition 5. Let $f(x)$ be a function. We say

$$\lim_{x \to a^-} f(x) = -\infty$$

if for every (negative) number M there is a corresponding $\delta > 0$ such that $f(x) < M$ whenever $a - \delta < x < a$.

Definition 6. Let $f(x)$ be a function. We say

$$\lim_{x \to a^+} f(x) = -\infty$$

if for every (negative) number M there is a corresponding $\delta > 0$ such that $f(x) < M$ whenever $a < x < a + \delta$.

Definition 7. Let $f(x)$ be a function. We say

$$\lim_{x \to \infty} f(x) = L$$

if for every $\epsilon > 0$ there is a corresponding number N such that $|f(x) - L| < \epsilon$ whenever $x > N$.

Definition 8. Let $f(x)$ be a function. We say

$$\lim_{x \to -\infty} f(x) = L$$

if for every $\epsilon > 0$ there is a corresponding N such that $|f(x) - L| < \epsilon$ whenever $x < N$.

Definition 9. Let $f(x)$ be a function. We say

$$\lim_{x\to\infty} f(x) = \infty$$

if for every (positive) number M there is a corresponding number N such that $f(x) > M$ whenever $x > N$.

Definition 10. Let $f(x)$ be a function. We say

$$\lim_{x\to\infty} f(x) = -\infty$$

if for every (negative) number M there is a corresponding number N such that $f(x) < M$ whenever $x > N$.

Definition 11. Let $f(x)$ be a function. We say

$$\lim_{x\to-\infty} f(x) = \infty$$

if for every (positive) number M there is a corresponding number N such that $f(x) > M$ whenever $x < N$.

Definition 12. Let $f(x)$ be a function. We say

$$\lim_{x\to-\infty} f(x) = -\infty$$

if for every (negative) number M there is a corresponding number N such that $f(x) < M$ whenever $x < N$.

Note that whenever the above definitions say (positive) or (negative) for M, it actually is not necessary to restrict M to be positive or negative. It is only there to help you understand the role of M in the definition. Another reason it is there in the definition is that it may simplify a particular argument that uses one of these definitions if M is assumed to be negative or positive only.