Notes for Section 2.4

We have already talked about the Product Rule and have seen how to use it on several examples. I do not expect that you will have much difficulty in learning how to use it. Take a look at Examples 1-3 in Section 2.4 and let me know if you have questions. Here is an example for how the Product Rule can be used to differentiate a product of three functions:

Example 1. *Let us differentiate* $f(x) = x^2 \sin^2(x)$.

By the Product Rule,

$$\begin{aligned} \frac{d}{dx}[x^2 \sin^2(x)] &= \frac{d}{dx} x^2 \sin^2(x) + x^2 \frac{d}{dx} \sin^2(x) \\ &= 2x \sin^2(x) + x^2 \frac{d}{dx} \sin^2(x) \end{aligned}$$

Notice the use of the brackets above. This is because by convention, $\frac{d}{dx}$ applies only the function that immediately follows it. So in $\frac{d}{dx} x^2 \sin^2(x)$ without brackets on the right-hand side, we only differentiate x^2 and not $x^2 \sin^2(x)$. Anyway, we need to deal with $\frac{d}{dx} \sin^2(x)$. We can do this by using the Product Rule again:

$$\begin{aligned} \frac{d}{dx} \sin^2(x) &= \frac{d}{dx}[\sin(x) \sin(x)] \\ &= \frac{d}{dx} \sin(x) \sin(x) + \sin(x) \frac{d}{dx} \sin(x) \\ &= \cos(x) \sin(x) + \sin(x) \cos(x) \\ &= 2 \sin(x) \cos(x) \end{aligned}$$

So

$$\frac{d}{dx}[x^2 \sin^2(x)] = 2x \sin^2(x) + 2x^2 \sin(x) \cos(x).$$

Now, you can try to differentiate $f(x) = x^2 \sin^2(x)$ a different way for practice. Choose $g(x) = x^2 \sin(x)$ and $h(x) = \sin(x)$ and use the Product Rule to differentiate $g(x)h(x)$ first, then use the Product Rule again to tackle $\frac{dg}{dx}$.

We owe the proof of the Product Rule. The one given in the book is fine, but not well motivated. Let me give a similar argument, which is somewhat longer, but perhaps easier to understand. So let f and g be functions that are differentiable at x. Let us start off by writing

$$\Delta f = f(x+h) - f(x) \qquad \text{and} \qquad \Delta g = g(x+h) - g(x).$$

So

$$f'(x) = \lim_{h \to 0} \frac{\Delta f}{h} \qquad \text{and} \qquad g'(x) = \lim_{h \to 0} \frac{\Delta g}{h}.$$

Now

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

If we substitute

$$f(x+h) = f(x) + \Delta f \qquad \text{and} \qquad g(x+h) = g(x) + \Delta g$$

we get

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h \to 0} \frac{(f(x) + \Delta f)(g(x) + \Delta g) - f(x)g(x)}{h}.$$

Let us work on the expression in the numerator a bit:

$$\begin{aligned} (f(x) + \Delta f)(g(x) + \Delta g) - f(x)g(x) &= f(x)g(x) + f(x)\Delta g + g(x)\Delta f + \Delta f \Delta g - f(x)g(x) \\ &= f(x)\Delta g + g(x)\Delta f + \Delta f \Delta g \end{aligned}$$

So

$$\begin{aligned}\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x)\Delta g + g(x)\Delta f + \Delta f\Delta g}{h}\\ &= \lim_{h\to 0}\frac{f(x)\Delta g}{h} + \lim_{h\to 0}\frac{g(x)\Delta f}{h} + \lim_{h\to 0}\frac{\Delta f\Delta g}{h}\end{aligned}$$

by LL1. Now by LL3,

$$\lim_{h\to 0}\frac{f(x)\Delta g}{h} = f(x)\lim_{h\to 0}\frac{\Delta g}{h} = f(x)g'(x),$$

and

$$\lim_{h\to 0}\frac{g(x)\Delta f}{h} = g(x)\lim_{h\to 0}\frac{\Delta f}{h} = g(x)f'(x).$$

By LL4,

$$\lim_{h\to 0}\frac{\Delta f\Delta g}{h} = \left[\lim_{h\to 0}\frac{\Delta f}{h}\right]\left[\lim_{h\to 0}\Delta g\right] = f'(x)\lim_{h\to 0}\Delta g.$$

Now

$$\lim_{h\to 0}\Delta g = \lim_{h\to 0}[g(x+h) - g(x)] = \left[\lim_{h\to 0} g(x+h)\right] - g(x)$$

by LL2 and LL7 as $g(x)$ does not depend on h. Since g is differentiable at x, it is continuous at x by Theorem 4 in Section 2.2. So we can use direct substitution in

$$\lim_{h\to 0} g(x+h) = g(x+0) = g(x).$$

Hence

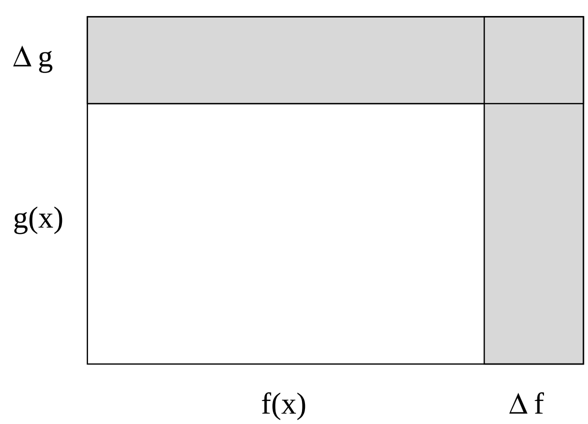
$$\lim_{h\to 0}\Delta g = g(x) - g(x) = 0.$$

Let us put the pieces together:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x) + f'(x)\cdot 0 = f(x)g'(x) + f'(x)g(x),$$

which is exactly the Product Rule.

What we just did here can be visualized using some geometry. Suppose for now that $f(x)$, $g(x)$, Δf, and Δg are all positive. Then $f(x+h)g(x+h) - f(x)g(x)$ is the area of the shaded L-shaped region in the following diagram:



As $h \to 0$, the two long rectangles become really skinny, while the rectangle that represents $\Delta f\Delta g$ becomes tiny even relative to those two skinny rectangles because both its width and height approach 0. This is why the numerator $f(x+h)g(x+h) - f(x)g(x)$ becomes $f(x)\Delta g + g(x)\Delta f$ in the limit, and this is why the Product Rule says $(fg)' = f'g + fg'$ and not $(fg)' = f'g'$.

Let us now deal with quotients. If f and g are both differentiable functions at x and $g(x) \neq 0$, then

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Skip the proof for now, and study Examples 4 and 5 in Section 2.4. Once you are comfortable with how the Quotient Rule is used in those examples, make up an example of a quotient from functions you know how to differentiate (i.e. power functions, root functions, sin and cos) and differentiate it. Note that unlike the Product Rule, the Quotient Rule is not symmetric in f and g because while $fg = gf$, $\frac{f}{g} \neq \frac{g}{f}$.

The Quotient Rule can be proved using a similar, but slightly more complicated argument as the Product Rule. You can try to do this the way we handled the proof of the Product Rule above, or you can try to follow the argument given in the book. But don't stress too much about either. We will soon see an easier way to prove and even remember the Quotient Rule.

Now that we have the Quotient Rule, we can easily find the derivatives of tan, sec, csc, and cot using

$$\begin{aligned}
\frac{d}{dx}\tan(x) &= \frac{d}{dx}\frac{\sin(x)}{\cos(x)}\\
\frac{d}{dx}\sec(x) &= \frac{d}{dx}\frac{1}{\cos(x)}\\
\frac{d}{dx}\csc(x) &= \frac{d}{dx}\frac{1}{\sin(x)}\\
\frac{d}{dx}\cot(x) &= \frac{d}{dx}\frac{\cos(x)}{\sin(x)}.
\end{aligned}$$

They are easy enough to do and I will leave the details to you. The book does the first one, but I'd say you should try doing it yourself instead of reading it. The results are given in the book, you can check your work against those.