Notes for Section 2.6

This section is on an application of the Chain Rule. It shows you how to find the slopes of curves which are not necessarily graphs of functions or at least are not expressed in the form $y = f(x)$. The introductory example of the circle $x^2 + y^2 = 25$ in the book is a perfectly good example, so let's look at that. Here is a link to desmos with that circle and the tangent line to that circle at the point $(3, 4)$. It is clear from the graph that there is a tangent line at $(3, 4)$ and that the circle has a certain slope, the same slope as the tangent line's at that point. The question is how to find that slope. One approach is to solve the equation $x^2 + y^2 = 25$ for y in terms of x:

$$\begin{aligned} x^2 + y^2 &= 25 \\ y^2 &= 25 - x^2 \\ \sqrt{y^2} &= \sqrt{25 - x^2} \\ |y| &= \sqrt{25 - x^2} \\ y &= \pm\sqrt{25 - x^2} \end{aligned}$$

We get the $\pm$ because for every value of $x \in (-5, 5)$, there are two corresponding points on the circle, one above and one below the x-axis, with two different values of y. This is exactly why $x^2 + y^2 = 25$ is a relation between x and y but does not make y a function x: there are multiple values of y that correspond to the same value of x. If $x = 3$, then $y = \pm 4$. Of these, we are interested in the positive value. If you look at just a small part of the circle near the point $(3, 4)$, you will see that $y = \sqrt{25 - x^2}$ for all those points. In other words, the part of the circle that is below the x-axis does not matter if we want to find the tangent line at $(3, 4)$. A fancier way to say this is that y is locally a function of x in a sufficiently small neighborhood of $(3, 4)$ or that y is an implicit function of x. So for our purposes, $y(x) = \sqrt{25 - x^2}$, and this is indeed a function. We know how to differentiate it using the Chain Rule:

$$\frac{dy}{dx} = \frac{d}{dx} 25 - x^2 = \frac{d}{dx}(25 - x^2)^{1/2} = \frac{1}{2}(25 - x^2)^{-1/2}(-2x) = -\frac{x}{\sqrt{25 - x^2}}.$$

So $y'(3) = -3/\sqrt{25 - 3^2} = -\frac{3}{4}$. Now, if you want the equation of the tangent line, you do the usual thing to write down the equation of a line with slope $-3/4$ that passes through $(3, 4)$. I will leave the details to you.

There is an easier way to do this. Let us treat y as a local function $y = y(x)$ and differentiate $x^2 + y^2$:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(x^2 + [y(x)]^2) = 2x + 2y(x)\frac{dy}{dx} = 2x + 2y\frac{dy}{dx}.$$

The reason for the $2y\frac{dy}{dx}$ part is that $[y(x)]^2$ is a composite function and we had to use the Chain Rule to differentiate it. Now, we also know that $x^2 + y^2 = 25$, so

$$\begin{aligned} \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx} 25 \\ 2x + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{2y} = -\frac{x}{y} \end{aligned}$$

At $(3, 4)$, $\frac{dy}{dx} = -\frac{3}{4}$. This is called implicit differentiation. The basic idea is to differentiate both sides of the equation treating one of the variables as a (local or implicit) function of the other. Do not forget to use the Chain Rule. You can then solve the resulting equation for the derivative you want to find. The equation that you need to solve for the derivative is always a linear equation, so it is always quite straightforward to solve.

The second method of finding the slope is not only more straightforward, often it is the only way to do it. While the equation $x^2 + y^2 = 25$ is easy enough to solve for y, there may not be good ways to express y in terms of x from other equations. For example, the equation $\sin(x+y) = y^2\cos(x)$ of the curve in Example 3 cannot easily be solved for y, but the slope can be found using implicit differentiation. Look at Examples 2-4. Example 4 also shows you how you can use implicit differentiation to find higher derivatives.

So is there anything that can go wrong with this process? Suppose we had the find the slope $\frac{dy}{dx}$ of $x^2 + y^2 = 25$ at the point $(5, 0)$. We still find $\frac{dy}{dx} = -\frac{x}{y}$, except now $y = 0$, so $-\frac{x}{y}$ does not have a value. This makes sense if you look at the graph of the circle again. The tangent line at $(5, 0)$ is vertical, so its slope is not defined. Another thing you may notice is that no matter how small a neighborhood of $(5, 0)$ you look at, the part of the circle that is inside that neighborhood can never be the graph of a function because it does not pass the vertical line test. In other words, y is not an implicit (or local) function of x near $(5, 0)$. In Example 2,

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

Notice that this gives $0/0$ at the point $(0, 0)$. In fact, the curve–graph it in desmos!–intersects itself at that point. Again, if you look at a neighborhood of $(0, 0)$, no matter how small, the part of the graph in that neighborhood can never be the graph of a function because it always fails the vertical line test. So y is not an implicit function of x near $(0, 0)$.