Notes for Section 2.7

Related rates are yet another practical application of the Chain Rule. Read and understand Example 1 about a balloon (sphere) whose radius and hence volume are changing over time. The example shows you how to use the standard relationship $V = \frac{4}{3}\pi r^3$ between the volume and the radius to relate the two rates of change to each other. The thing to keep in mind here is that V and r are really functions $V(t)$ and $r(t)$. So both sides of

$$V(t) = \frac{4}{3}\pi[r(t)]^3$$

are functions of t, and since they are equal, their rates of change with respect to t must also be equal. This is why we can differentiate both sides of the equation. The part $[r(t)]^3$ is a composite function, so the Chain Rule must be used. Now, if we know the current radius r and the rate of change of the volume $\frac{dV}{dt}$ then we can solve the equation for the rate of change of the radius $\frac{dr}{dt}$. The details are in the example.

Study Examples 2-5 to further familiarize yourself with the process. The biggest challenge in related rate problems tends to be turning a word problem into equations. As with word problems in general, this requires practice. There is no secret recipe that tells you how to solve every problem. If there were one, we would have told you by now. I can give you the typical general advice for solving word problems:

- Read the problem carefully and make sure you understand what it says and what it is asking for. Visualize the situation.
- Identify variables and give them names. Choose appropriate units. This is a crucial (and often overlooked) step and students often set themselves up for failure here by not being precise about what a variables measure. E.g. in Example 4, writing things like

$$\begin{aligned} x &= \text{Car A} \\ y &= \text{Car B} \end{aligned}$$

  is not meaningful. For one, Car A and Car B are not numerical quantities. Chances are you mean that x and y are measures of some quantitative parameters of Cars A and B, but it is unclear what numerical parameter. There are many quantities one can associate with a car, such as its length, weight, speed, position, the driver's license number, etc. Carefully identifying the ones that are relevant to the problem is an important part of successful problem solving. If you are not careful in your thinking and writing, chances are you will confuse yourself later on and will fail to solve the problem correctly. In this case, saying

$$\begin{aligned} x &= \text{Car A's distance from the intersection at time } t \\ y &= \text{Car B's distance from the intersection at time } t \end{aligned}$$

  is a big improvement over what is above. Even better would be to pay some attention to physical units and specify them in defining the variables, as in

$$\begin{aligned} t &= \text{time in hours} \\ x(t) &= \text{Car A's distance in miles from the intersection at time } t \\ y(t) &= \text{Car B's distance in miles from the intersection at time } t \end{aligned}$$

- Find the relevant relationships among the variables and (carefully and precisely) express them as equations, functions, etc. You will find that this is easier to do and more likely to be correct if you did a good job at identifying the variables.
- Do the math. Manipulate the equations, functions, etc that you set up until you find what the problem asked you to find. In the case of related rate problems, you would be differentiating an equation that involves functions to find a relationship between rates of change. This

is often easier to do than setting up the problem, but requires attention to detail. For example, you will want to keep careful track of which is the independent variable and which are the dependent variables (functions) that depend on it. Otherwise you may get confused about which variable you are differentiating with respect to. Also, remember that composite functions are common in related rate problems and you need to use the Chain Rule when differentiating them.

- Another often overlooked step is checking your answer. Does the result make intuitive sense? Does it have the units you expect and do those units follow from the calculations you did? Checking units is usually a fairly quick and easy thing to do, and if the units you get from your calculations are not what you expect, it is not only a good indication that your work is incorrect but it can often give a hint about what kind of mistake you made. There is always the possibility of substituting your results back into your equations, but this would not catch any mistakes you may have made in setting up those equations. So it is better to see if the result fits the conditions of the original word problem.

Since there is no particular theory to related rates, and they are all about working examples, I would be glad to work through an example with you in class. Let me know if you have a particular one that you would like to do. I like exercise 2.7.23, but will be glad to do whatever you like.