Notes for Section 4.2

In the last section you learned how to find the area under the graph of a function by approximating it with a sum of areas of rectangles and then taking the limit as the widths of the rectangles approach 0. In this Section 4.2, you learn more about such sums of rectangles and the area under a graph.

First, recall that the area under the graph of a sufficiently well-behaved function $f(x)$ over the interval $[a, b]$ is the limit

$$\lim_{\max(x_{i+1}-x_i)\to 0} \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$

where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

and

$$x_1^* \in [x_0, x_1], x_2^* \in [x_1, x_2], \ldots, x_n^* \in [x_{n-1}, x_n].$$

The points $x_0, x_1, \ldots, x_n$ divide the interval $[a, b]$ into a collection of smaller subintervals

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$$

called a *partition* of $[a, b]$. We called $x_1^*, x_2^*, \ldots, x_n^*$ the sample points, as they are the points where we sample the values of f. The sum

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$

is called a *Riemann sum.* We can make the notation a little slicker by letting $\Delta x_i = x_i - x_{i-1}$ be the width of the i-th subinterval $[x_{i-1}, x_i]$ and writing the Riemann sum as

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

Figure 1 shows the relationship between the $x_0, \ldots, x_n$, the $x_1^*, \ldots, x_n^*$, and the $\Delta x_1, \ldots, \Delta x_n$ on the number line.

We have talked about partitions whose subintervals have uniform widths. Such a partition is called a *regular partition* or a *uniform partition.* In this case $\Delta x_1 = \Delta x_2 = \cdots = \Delta x_n$. So we can just refer to all of them as Δx and then the Riemann sum has the somewhat simpler form

$$\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

So far all the examples we looked at were areas under the graphs of functions whose values are positive. The Riemann sum is the same even if f has some negative values. But the geometric picture is a little different. In that case, some of the of the products $f(x_i^*)\Delta x_i$ in the Riemann sum have negative values. The corresponding rectangles are below the x-axis and their areas count negatively toward the Riemann sum. So the Riemann sum is not so much measuring the area under the graph of f between $x = a$ and $x = b$, but the area between the graph of f and the x-axis, and any area that is actually below the x-axis counts as negative area. The area is still often referred to as the area under the graph or to be a little more precise, the signed area under the graph, only because it is awkward to have to spell out everytime that we mean the area between the x-axis and the graph with the parts under the x-axis counted as negative areas. To avoid always having the write this out in so many words, we have a name and convenient notation for it: it is the definite integral (or just integral) of f from a to b and it is denoted by

$$\int_a^b f(x)\,dx.$$

By definition (see Definition 2),

$$\int_a^b f(x)\,dx = \lim_{\max(\Delta x_i)\to 0} \sum_{i=1}^n f(x_i^*)\Delta x_i.$$

Of course, this limit may or may not exist. As I mentioned in the notes for Section 4.1, for this limit to exist, not only does the Riemann sum need to get arbitrarily close to some number as $\max(\Delta x_i) \to 0$, but it has to do so independently of the choice of partitions and sample points along the way. A function f is called *integrable* on the interval $[a, b]$ if the the limit of the Riemann sum in the definition of $\int_a^b f(x)\,dx$ exists. Otherwise, f is *nonintegrable* and its definite integral from a to b is not defined.

The integral sign $\int$ is similar to a capital S as a reminder of the Greek capital letter sigma Σ in the Riemann sum. The dx is a reminder of the Δx term in the Riemann as sum as $\Delta x \to 0$, just like the dx in $\frac{df}{dx}$ is a reminder of the Δx in the denominator of the difference quotient. That dx part is an important part of the notation. It is what tells you what the independent variable is. E.g. if we write

$$\int_0^1 x^t\,dx$$

we mean the area under the graph the function $f(x) = x^t$, where t is a constant parameter. If we only wrote $\int_0^1 x^t$ it would be unclear whether x is the variable and t is constant, or the other way, or perhaps both x and t are variables. So do not forget to include it in your notation. That said, it does not matter what letter we use to denote the independent variable. The function $f(x) = x^2$ is the same as the function $f(t) = t^2$. They have the same graph too, only in one case we label the horizontal axis by x and in the other case we label it by t. Therefore the area under the graph from a to b is the same too:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

Of course, we could have used any other letter for the independent variable.

It is hard to describe the exact conditions that make a function well-behaved enough to be integrable over an interval $[a, b]$, other than to say that the limit of the Riemann sum exists. But it is not so hard to give sufficient conditions for integrability. Theorem 3 does exactly that and tells us that a function f that is continuous over $[a, b]$ is certainly integrable on $[a, b]$. In fact, f can have some discontinuities on $[a, b]$: as long as they are all jump discontinuities and there are finitely many of them, f will still be integrable on $[a, b]$. We are not saying that functions that have infinitely many discontinuities or have some discontinuities that are not jump discontinuities are certainly not integrable. Some such functions are, and some are not.

So how can we calculate the definite integral of a function? At this point, the only thing we know is that we can try to find the limit of the Riemann sum. But this is not an easy thing to do in general, since the value of the sum itself depends on the choice of partition and sample points. However, if you know that f is integrable on $[a, b]$ (e.g. because f is continuous), then the limit of any Riemann sum should be the definite integral. So you could use the left-hand sum with uniform partititons, or the right-hand sum, or the midpoint sum, or the lower sum, or the upper sum, whichever is convenient. This is what Theorem 4 says, speficifically for the right-hand sum with uniform partitions.

Being able to go back and forth between definite integrals and Riemann sums is a useful skill and is illustrated by Example 1. Example 2 shows you how you can find the definite integral by actually evaluating the limit of a Riemann sum in a simple case. Example 3 shows that you can sometimes find the definite integral without using limits or doing any calculus at all just be interpreting it in terms of geometric shapes whose areas you know how to calculate.

We already talked about the midpoint sum or midpoint rule in the last section. This section has an example (Example 4) of how it can be calculated, at least using a finite number of rectangles.

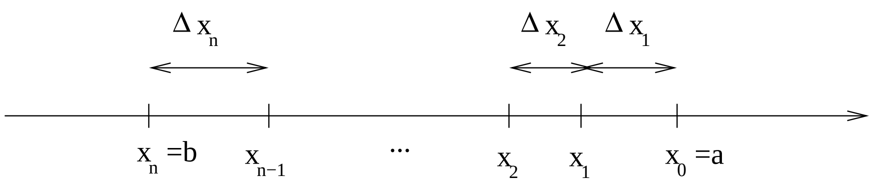
The rest of this section introduces a few useful properties of definite integrals. The first one may seem a little strange. So far, we always assumed $a < b$ when we talked about areas under the graph of f between a and b. But we can make sense of the Riemann sum even if $b < a$. Notice that in this case there is no interval $[a, b]$, but we can view the graph of f over the interval $[b, a]$. Only now $x_0, x_1, \ldots, x_n$ must satisfy

$$a = x_0 > x_1 > x_2 > \cdots > x_n = b$$

and $[b, a]$ is partitioned into subintervals

$$[x_n, x_{n-1}], [x_{n-1}, x_{n-2}], \ldots, [x_2, x_1], [x_1, x_0].$$

Here is what this looks like on the number line:



The sample points are still points within these subintervals, only now $x_i^* \in [x_i, x_{i-1}]$. The Riemann sum is the same:

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}).$$

But $x_i - x_{i-1}$ is now negative. So those rectangles that are above the x-axis count as negative ones and those below the x-axis as positive ones. When we take the limit as all the $x_i - x_{i-1} \to 0$, we get the signed area between x-axis and the graph, with the area above the x-axis counting as negative area and the area below the x-axis as positive area. Hence

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

That

$$\int_a^a f(x)\,dx = 0$$

should make sense as this time we are looking at the signed area between the x-axis and the graph of f over an interval of width 0.

On to the Properties 1-4. As the textbook explains, Property 1 for the definite integral of the constant function $f(x) = c$ follows from finding the area of just one rectangle, see Figure 14. Properties 2-4 follow from some straighforward algebra and the corresponding limit laws. For Property 2, note

$$\begin{aligned}\sum_{i=1}^{n} cf(x_i^*)\Delta x_i &= cf(x_1^*)\Delta x_1 + cf(x_2^*)\Delta x_2 + \cdots + cf(x_n^*)\Delta x_n \\ &= c(f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n) \\ &= c\sum_{i=1}^{n} f(x_i^*)\Delta x_i\end{aligned}$$

and use Limit Law 3. For Property 3, note

$$\begin{aligned}
\sum_{i=1}^{n}[f(x_i^*)+g(x_i^*)]\Delta x_i &= [f(x_1^*)+g(x_1^*)]\Delta x_1 + [f(x_2^*)+g(x_2^*)]\Delta x_2 + \cdots + [f(x_n^*)+g(x_n^*)]\Delta x_n \\
&= f(x_1^*)\Delta x_1 + g(x_1^*)\Delta x_1 + \cdots + f(x_n^*)\Delta x_n + g(x_n^*)\Delta x_n \\
&= f(x_1^*)\Delta x_1 + \cdots + f(x_n^*)\Delta x_n + g(x_1^*)\Delta x_1 + \cdots + g(x_n^*)\Delta x_n \\
&= \sum_{i=1}^{n} f(x_i^*)\Delta x_i + \sum_{i=1}^{n} g(x_i^*)\Delta x_i
\end{aligned}$$

and use Limit Law 1. Property 4 follows similarly using Limit Law 2.

Properties 5-8 should make good intuitive sense if you consider what they mean in terms of areas under graphs. It is not hard to show that they are true using Riemann sums either, but I am certain that studying Figure 16-18 will help you understand these properties more than an algebraic argument about Riemann sums. There is no figure illustrating Property 6. Can you make your own? Or can you use the Riemann sum to explain why a function whose values are nonnegative must have a definite integral that is also nonnegative?

We will soon learn a more efficient way to evaluate definite integrals than limits of Riemann sums.