Notes for Section 4.3

In Section 3.7, we learned about how to find the displacement of an object moving along a straight path from its velocity $v(t)$ between two points in time $t = a$ and $t = b$ by taking the antiderivative of the function v. In Sections 4.1 and 4.2, we looked at the same problem from a different angle and approximated the displacement by the Riemann sum of the velocity over a partition of the interval $t \in [a, b]$. We also noted that the if we take the limit of the Riemann sum as the widths of the subintervals approach 0, then we get the exact displacement of the moving object between $t = a$ and $t = b$. We found that the limit of the Riemann sum is the area under the graph of $v(t)$ between $t = a$ and $t = b$ and called that area the definite integral $\int_a^b v(t)\,dt$. If you put the pieces together, it will not come as a surprise that the definite integral of $v(t)$ has something to do with the antiderivative of $v(t)$.

In fact, we can say much more about the relationship between the two. Suppose that an object moves at velocity $v(t)$ at time t. How much distance does that object cover between $t = a$ and $t = b$? We know that the answer is $\int_a^b v(t)\,dt$. But we also know that if the position of the object is $s(t)$ at time t, then s is an antiderivative of v. But there are infinitely many antidervatives of the same function v because the antiderivative can only be recovered from the function v up to a constant. So which of the infinitely many antiderivatives is s? Actually, that does not matter because we are not really asking for the position of the object at time t. We are asking for the displacement of the object between $t = a$ and $t = b$, that is the difference $s(b) - s(a)$ between its positions at time $t = a$ and $t = b$. And that does not depend on the choice of s. Here is why. Suppose r is another antiderivative of v. Then s and r differ by some constant c. (Since $r'(t) = s'(t) = v(t)$ on $[a, b]$, we know $r(t) = s(t) + c$ by Corollary 7 in Section 3.2.) So

$$r(b) - r(a) = (s(b) + c) - (s(a) + c) = s(b) + c - s(a) - c = r(b) - r(a).$$

This shows that any of the antiderivatives of v can be used to calculate the displacement. As we remarked above, the displacement is also $\int_a^b v(t)\,dt$. So it should be true that

$$\int_a^b v(t)\,dt = s(b) - s(a).$$

The argument I just gave is not exactly a rigorous mathematical argument. For example, I ignored the question of whether the definite integral even exists. But it does at least give us a very strong hint that there may be an easy way to calculate the definite integral of a function by using an antiderivative of that function. In fact, while I have so far talked about velocity and position at time t, really the only thing that mattered about these was the fact the velocity is the rate of change of position. So everything I have said should work the same for any function $f(x)$ and any one of its antiderivatives $F(x)$, giving us the result

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Let's test this on an example. In Example 2 in Section 4.1, the area under the graph of $f(x) = x^2$ between $x = 0$ and $x = 1$ was calculated using the limit of the right-hand sum. It was found to be 1/3. So

$$\int_0^1 x^2\,dx = \frac{1}{3}.$$

We know that the general antiderivative of $f(x) = x^2$ is $\frac{x^3}{3} + c$. We only need one of these antiderivatives, say $F(x) = \frac{x^3}{3}$. Now

$$F(1) - F(0) = \frac{1}{3} - \frac{0}{3} = \frac{1}{3}.$$

That's a match! So indeed

$$\int_0^1 x^2\,dx = F(1) - F(0).$$

Another definite integral whose value we know is

$$\int_0^3 x^3 - 6x\,dx = -\frac{27}{4}$$

from Example 2 in Section 4.2. It is easy enough to figure out that the antiderivative of $x^3 - 6x$ is $\frac{x^4}{4} - 6\frac{x^2}{2} + c = \frac{x^4}{4} - 3x^2 + c$. Since we can just use $F(x) = \frac{x^4}{4} - 3x^2$ and calculate

$$F(3) - F(0) = \frac{3^4}{4} - 3(3^2) - \left(\frac{0^4}{4} - 3(0^2)\right) = \frac{81}{4} - 27 = \frac{81}{4} - \frac{108}{4} = -\frac{27}{4}.$$

Success again. In fact, what we have found is true in general and our textbook calls it the Evaluation Theorem:

Theorem (The Evaluation Theorem)**.** *Let f be a function that is continuous on the closed interval $[a, b]$. If F is an antiderivative of f on $[a, b]$, that is $F' = f$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

We have already seen two examples of this. The textbook has another two in Examples 1 and 2. Study those first and understand how helpful this theorem is in evaluating definite integals. It sure beats having to find the limit of a Riemann sum.

The reason we stated the theorem for continuous functions is that we know for sure that the definite integral exists if f is continuous. Incidentally, finding the antiderivative of a function that has discontinuities can be tricky too. But evaluating the definite integral of a function that has only finitely many discontinuities may not be that hard by dividing it up into continuous pieces and dealing with the pieces one at a time.

I will give you my version of the proof of the Evaluation Theorem, although it is the same as the argument in the textbook. So suppose f is continuous on $[a, b]$ and F is such that $F' = f$ on $[a, b]$. Since f is continuous, the definite integral

$$\int_a^b f(x)\,dx$$

exists by Theorem 3 in Section 4.2 and it is the limit of any Riemann sum of f over $[a, b]$. Let $x_0 = a, x_1, x_2, \ldots, x_n = b$ be a uniform partition of $[a, b]$. So $\Delta x = \frac{b-a}{n}$ is the width of each subinterval and

$$x_i = a + i\Delta x.$$

We will now choose the sample points $x_1^*, x_2^*, \ldots, x_n^*$ in a very clever way. Notice that the antiderivative F is a differentiable function on $[a, b]$ since its derivative is f. Therefore F is also continuous on $[a, b]$. In particular, F is also differentiable and continuous on each subinterval $[x_{i-1}, x_i]$. Hence the Mean Value Theorem applies to F and says that there is exists a point $c_i \in [x_{i-1}, x_i]$ such that

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = \frac{F(x_i) - F(x_{i-1})}{\Delta x}.$$

Let the sample point $x_i^* = c_i$. Now

$$f(x_i^*) = F'(x_i^*) = \frac{F(x_i) - F(x_{i-1})}{\Delta x} \implies f(x_i^*)\Delta x = F(x_i) - F(x_{i-1}).$$

Hence the Riemann sum is

$$\begin{aligned}\sum_{i=1}^{n} f(x_i^*)\Delta x &= \sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1})\big) \\ &= F(x_1) - F(x_0) + F(x_2) - F(x_1) + \cdots + F(x_{n-1}) - F(x_{n-2} + F(x_n) - F(x_{n-1}) \\ &= F(x_n) - F(x_0) \\ &= F(b) - F(a)\end{aligned}$$

as most of the terms actually cancel. By the way, such a sum is called a telescoping sum. I think you can guess why. Since all of the Riemann sums are equal to the same constant number $F(b) - F(a)$, their limit is also $F(b) - F(a)$. Hence

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x = F(b) - F(a).$$

Now, remember when we learned the Mean Value Theorem, I told you that it was not a practical, computational result, but more of a theoretical tool whose importance was in constructing arguments. You have just seen how the MVT helped us construct the argument to justify the Evaluation Theorem. The Evaluation Theorem is a really useful practical and computational tool. It is also an important theoretical result, which can be used to construct other arguments.

As the difference $F(b) - F(a)$ shows up so often in evaluating definite integrals, there are various shorthand notations for it, such as

$$F(b) - F(a) = [F(x)]_a^b = F(x)]_a^b = F(x)|_a^b\,.$$

Because of their connection to definite integrals, antiderivatives are also called *indefinite integrals* and the general antiderivative of f is denoted by

$$\int f(x)\,dx.$$

So for example

$$\int \cos(x)\,dx = \sin(x) + c.$$

There is nothing new here, other than the name and the notation. Do not confuse definite and indefinite integrals: a definite integral of f is a number, the area under a graph of f, whereas the indefinite integral of f is a family of functions, the family of all antiderivatives of f. Using the indefinite integral notation, the Evaluation Theorem says

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big]_a^b$$

Do not worry too much about the fact that $\int f(x)\,dx$ is technically not one function, but an infinite family of functions when evaluating it at a and b. It does not matter which of those functions you choose, they will all give you the same result. This is because the arbitrary constant c in the general antiderivative cancels anyway when you subtract $F(a)$ from $F(b)$. For example,

$$\begin{aligned}
\int_2^5 \frac{1}{x^2}\,dx &= \int \frac{1}{x^2}\,dx\Big]_2^5 \\
&= \int x^{-2}\,dx\Big]_2^5 \\
&= \left[\frac{x^{-1}}{-2} + c\right]_2^5 \\
&= \left[-\frac{1}{2x} + c\right]_2^5 \\
&= \left(-\frac{1}{4} + c\right) - \left(-\frac{1}{10} + c\right) \\
&= -\frac{1}{4} + c + \frac{1}{10} - c \\
&= -\frac{3}{20}
\end{aligned}$$

Example 3 reminds you how to do antiderivatives and Examples 4-6 further illustrate how you can use them to evaluate definite integrals. The section closes by listing a number examples of how the Evaluation Theorem can be applied in the sciences, including two more computational examples. The discussion preceding Example 7 and Example 7 itself remind me that I have been a little careless about using displacement and distance covered interchangeably. Technically, displacement is the difference in position between time $t = a$ and $t = b$ and while distance covered takes into account all of the changes in position with positive signs as your direction of motion might change. So for example, if you drive to the store at 1 PM and then come back home by 2 PM, your displacement between 1 PM and 2 PM is 0, since you are at home at both times, but the distance covered would be twice the distance between your home and the store. There is no difference if $v(t)$ is not negative, which was a tacit assumption as we calculated the distance covered over each subinterval of the partition. In general, velocity could of course be negative as well. Its integral still gives displacement, but if you want to calculate the distance covered, you need to make sure you take the absolute value of the change in position every time the velocity is negative. See Example 7 on how this can be done.