Notes for Section 4.4

In Section 4.3, we learned how to use the antiderivative of a function to find the area under the graph of that function. We will now turn this idea around and understand how the area under the graph can be used to find an antiderivative.

Let us start by revisiting the example of an object moving along a straight line at velocity $v(t)$. Suppose that the object is at position 0 at time $t = 0$. Can we find the position $s(x)$ of the object at time x? Sure, this is not a difficult question to answer. The displacement of the object between $t = 0$ and $t = x$ is the definite integral

$$s(x) - s(0) = \int_0^x v(t)\, dt.$$

The object started at position 0, so $s(0) = 0$ and hence

$$s(x) = \int_0^x v(t)\, dt.$$

We do know that position is one of the antiderivates of the velocity function, so s is an antiderivative of v. What we are saying is that $s'(x) = v(x)$.

What we have just observed may be true in general. Suppose f is a continuous function and that f has an antiderivative $F(x)$. Then

$$\int_0^x f(t)\, dt = F(x) - F(0)$$

by the Evaluation Theorem. Note that we used t as the independent variable in the integral because x is already in use as the upper bound of the definite integral. Both sides of this equation are functions of x. We can differentiate both sides and we get

$$\frac{d}{dx}\int_0^x f(t)\, dt = \frac{d}{dx}[F(x) - F(0)] = F'(x) - 0 = f(x).$$

This rounds out our picture of differentiation and integration as more or less inverse processes. What we have just discovered is true in greater generality too. Even if we do not know whether f has an antiderivative, we can still show that

$$\frac{d}{dx}\int_0^x f(t)\, dt = f(x)$$

if f is a continuous function. In fact, notice that there is nothing special about starting the definite integral at 0. The reason $\frac{d}{dx}F(0) = 0$ is not that the input to F is 0, but that $F(0)$ is a constant number and does not depend on x. So we could have chosen any other number a:

$$\frac{d}{dx}\int_a^x f(t)\, dt = \frac{d}{dx}[F(x) - F(a)] = F'(x) - 0 = f(x).$$

We can summarize what we have learned about the relationship between differentiation and integration in the following theorem.

Theorem (The Fundamental Theorem of Calculus)**.** *Let f be a continuous function on the interval $[a, b]$.*

1. *If F is an antiderivative of f on $[a, b]$, that is $F'(x) = f(x)$ at every $x \in [a, b]$, then*

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

2. *If the function* $g : [a, b] \to \mathbb{R}$ *is defined by*

$$g(x) = \int_a^x f(t)\,dt$$

then $g'(x) = f(x)$ *for every* $x \in [a, b]$.

The first half of this theorem is the Evaluation Theorem and we already proved it. Note that in the second half, the reason we know the function g exists is that we know that f is continuous, and hence the definite integral

$$\int_a^x f(t)\,dt$$

exists. This second half still requires proof. The argument we gave above works fine as long as we know that f has an antiderivative. The proof of the general case is given in the book. I cannot add much to it, so I will just let you read it.

Let us do a few examples on how to use it.

Example 1. *Let us find the derivative of the function*

$$g(x) = \int_0^x \cos(t^2)\,dt.$$

But the FTC,

$$g'(x) = f(x) = \cos(x^2).$$

So we are done. Notice that it is not easy to find an antiderivative of $\cos(x^2)$, at least in the sense that we could express it in terms of functions we are familiar with. But the function g defined above in terms of definite integral is an antiderivative of $\cos(x^2)$. So in some sense, it is easy to find an antiderivative of $\cos(x^2)$. The problem is that working with g is not easy. For example, how could we find the value of $g(1)$? We would have to evaluate

$$g(1) = \int_0^1 \cos(t^2)\,dt.$$

But we cannot use the Evaluation Theorem because the only antiderivative of $\cos(x^2)$ we know is g itself. So the only way we could try to find $g(1)$ is by taking the limit of Riemann sums.

If we want the general antiderivative of $\cos(x^2)$, we can say it is

$$\int_0^x \cos(t^2)\,dt + c.$$

Example 2. *What is the derivative of the function*

$$g(x) = \int_x^1 \sqrt[3]{1 + t^2}\,dt?$$

The trick here is to switch the bounds on the integral before using the FTC:

$$g(x) = \int_x^1 \sqrt[3]{t^2 + t + 1}\,dt = -\int_1^x \sqrt[3]{t^2 + t + 1}\,dt$$

by the property

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

So

$$g'(x) = \frac{d}{dx}\left[-\int_1^x \sqrt[3]{t^2 + t + 1}\,dt\right] = -\frac{d}{dx}\int_1^x \sqrt[3]{t^2 + t + 1}\,dt = -\sqrt[3]{x^2 + x + 1}$$

by the FTC.

Example 3. *What is the derivative of the function*

$$g(x) = \int_0^{x^2+1} \sin(\sqrt{t})\, dt?$$

The challenge is that the upper bound of the integral is now not just x. Let us define another function h by

$$h(x) = \int_0^{x} \sin(\sqrt{t})\, dt.$$

We know how to differentiate this one: $h'(x) = \sin(\sqrt{x})$ by the FTC. Notice that $g(x) = h(x^2+1)$, so

$$g'(x) = \frac{d}{dx} h(x^2+1) = h'(x^2+1)(2x)$$

by the Chain Rule. Since $h'(x^2+1) = \sin\left(\sqrt{x^2+1}\right)$, we get

$$g'(x) = 2x \sin\left(\sqrt{x^2+1}\right)$$

Example 4. *What is the derivative of the function*

$$g(x) = \int_x^{2x} \cos(t^2)\, dt?$$

Look $\cos(x^2)$ shows up again. The problem now is that both bounds of the integral depend on x. The trick is to write the integral as a difference

$$\int_x^{2x} \cos(t^2)\, dt = \int_0^{2x} \cos(t^2)\, dt - \int_0^{x} \cos(t^2)\, dt$$

Draw a diagram to convince yourself that this works. There is nothing special about 0, we could have put any other number instead, as long as it is in the domain of the function inside the integral. Now

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left[\int_0^{2x} \cos(t^2)\, dt - \int_0^{x} \cos(t^2)\, dt\right] \\ &= \frac{d}{dx}\int_0^{2x} \cos(t^2)\, dt - \frac{d}{dx}\int_0^{x} \cos(t^2)\, dt \end{aligned}$$

The second term is easy. By the FTC,

$$\frac{d}{dx}\int_0^{x} \cos(t^2)\, dt = \cos(x^2)$$

as we already figured out in one of the earlier examples. For the other term, we can use the Chain Rule just like in the last example. Let

$$h(x) = \frac{d}{dx}\int_0^{x} \cos(t^2)\, dt$$

Then $h'(x) = \cos(x^2)$.

$$\frac{d}{dx}\int_0^{2x} \cos(t^2)\, dt = \frac{d}{dx} h(2x) = 2h'(2x) = 2\cos(4x^2).$$

And so

$$g'(x) = \frac{d}{dx}\int_x^{2x} \cos(t^2)\, dt = 2\cos(4x^2) - \cos(x^2).$$

Let us think about that object moving along a straight line at velocity $v(t)$ at time t again. What is the average velocity between time $t = a$ and $t = b$? It is the displacement divided by the time elapsed. The displacement is

$$\int_a^b v(t)\, dt$$

and the time elapsed is $b - a$. So the average velocity is

$$v_{ave} = \frac{1}{b-a} \int_a^b v(t)\, dt$$

This example motivates the definition of the *average value of a function* f over the interval $[a, b]$ as

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

When we first enountered the Mean Value Theorem, I interpreted it for you in terms of the velocity of a moving car whose position at time t is $f(t)$. In any time interval $[a, b]$, there must be a point $c \in [a, b]$ such that instantaneous velocity $f'(c)$ is exactly the average velocity $\frac{f(b)-f(a)}{b-a}$. We can rephrase this in terms of integrals: there is a point $c \in [a, b]$ such that

$$v(c) = v_{ave} = \frac{1}{b-a} \int_a^b v(t)\, dt$$

as the right-hand side is the average velocity over the interval $[a, b]$. This also turns out to be true in general for any continuous function f.

Theorem (The Mean Value Theorem for Integrals)**.** *If f is a continuous function over the interval $[a, b]$, then there is a point $c \in [a, b]$ such that*

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx$$

or

$$\int_a^b f(x)\, dx = f(c)(b-a).$$

Another way you can look at this is that there is a point c where the height of the graph is such that a rectangle of that height and width $b - a$ has the same area as the area under the graph of f over $[a, b]$. The Mean Value Theorem for Integrals is a direct consequence of the FTC and the Mean Value Theorem for derivatives that we already know. The proof is short and straightforward and is in the book. Much like the Mean Value Theorem for derivatives, the Mean Value Theorem for Integrals is more of a theoretical tool than a computational one, since it does not tell us how to find the point c.