MCS 119 Exam 1 Solutions

1. (10 pts) Calculate the exact value of

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5}.$$

Make sure you carefully justify the steps in your calculation.

Since $x \to \infty$, x can be treated as a large positive number, and so we do not need to be concerned about what happens when $x = 0$. Therefore we can divide the numerator and the denominator by x:

$$\frac{\sqrt{2x^2+1}}{3x-5} = \frac{\frac{\sqrt{2x^2+1}}{x}}{\frac{3x-5}{x}} = \frac{\frac{\sqrt{2x^2+1}}{\sqrt{x^2}}}{3-\frac{5}{x}} = \frac{\sqrt{\frac{2x^2+1}{x^2}}}{3-\frac{5}{x}} = \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}},$$

where we used the fact that $x = \sqrt{x^2}$ since x is positive. We know that

$$\lim_{x\to\infty} \frac{1}{x^2} = 0,$$

and hence

$$\lim_{x\to\infty} \left(2 + \frac{1}{x^2}\right) = 2 + 0 = 2$$

by LL1 and LL7, and finally,

$$\lim_{x\to\infty} \sqrt{2+\frac{1}{x^2}} = \sqrt{\lim_{x\to\infty}\left(2+\frac{1}{x^2}\right)} = \sqrt{2}$$

by LL11. Similarly,

$$\lim_{x\to\infty} \frac{1}{x} = 0 \implies \lim_{x\to\infty}\left(3 - \frac{5}{x}\right) = \lim_{x\to\infty} 3 - 5\lim_{x\to\infty}\frac{1}{x} = 3 - 0 = 3$$

by LL2, LL3, and LL7. Therefore by LL5,

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \frac{\lim_{x\to\infty}\sqrt{2+\frac{1}{x^2}}}{\lim_{x\to\infty}\left(3-\frac{5}{x}\right)} = \frac{\sqrt{2}}{3}.$$

Here is another way this can be argued, which is less rigorous but is still convincing enough. If x is a large number, then

$$2x^2 + 1 \approx 2x^2 \quad \text{and} \quad 3x - 5 \approx 3x.$$

The relative error (compared to the size of the number) in these estimates gets smaller and smaller as x gets larger because the $2x^2$ and the $3x$ dominate compared to the 1 and the -5. So the estimates are getting better and better as $x \to \infty$. In fact, the estimates are perfect in the limit. Therefore

$$\frac{\sqrt{2x^2+1}}{3x-5} \approx \frac{\sqrt{2x^2}}{3x}$$

when x is large. This estimate also gets better and better as $x \to \infty$ and is perfect in the limit. Hence

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{2x^2}}{3x}.$$

Since $x \to \infty$, we can treat x as a (large) positive number. So

$$\sqrt{2x^2} = \sqrt{2}\sqrt{x^2} = \sqrt{2}|x| = \sqrt{2}x,$$

and

$$\frac{\sqrt{2x^2}}{3x} = \frac{\sqrt{2}x}{3x} = \frac{\sqrt{2}}{3}$$

because we can cancel x as $x \neq 0$. Finally, by LL7,

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{2}}{3} = \frac{\sqrt{2}}{3}$$

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2. (5 pts each) Let f be a function of real numbers and let a be a real number. State the informal and formal definitions of the following.

   (a) $\lim_{x\to a^-} f(x) = \infty$

   Informal definition: We say

   $$\lim_{x\to a^-} f(x) = \infty$$

   if $f(x)$ can be made arbitrarily large by making x sufficiently close to a (but not equal to to a).

   Formal definition: We say

   $$\lim_{x\to a^-} f(x) = \infty$$

   if for every (positive) number M there is a corresponding $\delta > 0$ such that $f(x) > M$ whenever $a - \delta < x < a$.

   (b) $\lim_{x\to\infty} f(x) = -\infty$

   Informal definition: We say

   $$\lim_{x\to\infty} f(x) = -\infty$$

   if $f(x)$ can be made arbitrarily negative by making x sufficiently large.

   Formal definition: We say

   $$\lim_{x\to\infty} f(x) = -\infty$$

   if for every (negative) number M there is a corresponding (positive) number N such that $f(x) < M$ whenever $x > N$.

3. (5 pts each) Let f be a function of real numbers such that f is continuous on the interval $[2,6]$, $f(2) = -1$, and $f(6) = 1$.

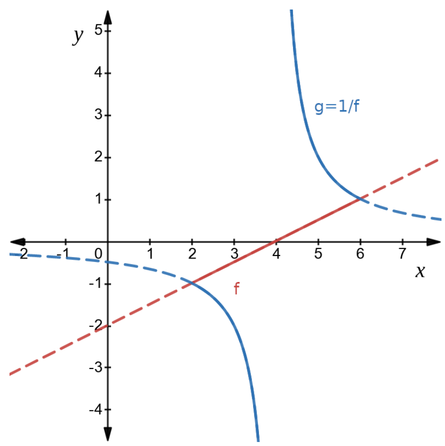
   (a) Below is an argument to show that there is a number c such that the value of $g(x) = 1/f(x)$ at $x = c$ is $1/2$. The argument has a mistake in it. Find it and explain why it is a mistake.
   Let $g(x) = 1/f(x)$. We know $f(2) = -1$ and $f(6) = 1$, therefore $g(2) = -1$ and $g(6) = 1$. Since f is continuous on the closed interval $[2,6]$, $g(x) = 1/f(x)$ is also continuous. The number $1/2$ is between -1 and 1, hence there must be a number $c \in (2,6)$ such that $g(c) = 1/2$ by the Intermediate Value Theorem. (Hint: You may find answering part (b) first easier. Or perhaps answering part (a) will lead you to the right idea for part (b). It all depends on how you think about it. But read both questions and see which one gives you an idea.)

   First, for $g(c)$ to be $1/2$, $f(c)$ must be 2. But 2 is not between $f(2) = -1$ and $f(6) = 1$. So there is no reason to believe that there is a number c between 2 and 6 that makes $f(c) = 2$. The mistake in the argument is the assertion that $g(x) = 1/f(x)$ is continuous.

Since $g(x)$ is the quotient of the constant function $h(x) = 1$ and the continuous function f, g must be continuous at every $x \in [2, 6]$ where $f(x) \neq 0$. But there could be some $x \in [2, 6]$ where $f(x) = 0$. Therefore g may not be continuous and the Intermediate Value Theorem is not applicable to g on the interval $[2, 6]$.
In fact, it is certain that g is not continuous on $[2, 6]$. Since f is continuous and 0 is between $f(2) = -1$ and $f(6) = 1$ the Intermediate Value Theorem guarantees that there is some $x \in (2, 6)$ where $f(x) = 0$, and g cannot be continuous there.

(b) Find an example of a function f that satisfies the conditions of this problem, yet there is no number $c \in (2, 6)$ such that that the value of $g(x) = 1/f(x)$ at $x = c$ is $1/2$. Justify your example.

All we need to do is construct a continuous function f such that $f(2) = -1$, $f(6) = 1$ and $f(c) \neq 2$ for any number $c \in (2, 6)$. A linear function $f(x) = mx + b$ through $(2, -1)$ and $(6, 1)$ will do as is quite clear from the graphs of f and g:



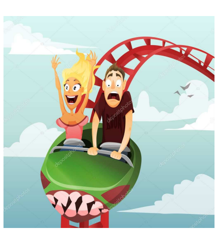
The slope is

$$m = \frac{1 - (-1)}{6 - 2} = \frac{2}{4} = \frac{1}{2}$$

and the y-intercept follows from

$$-1 = \frac{1}{2}2 + b \implies b = -2.$$

So $f(x) = x/2 - 2$ is a function that satisfies $f(2) = -1$, and $f(6) = 1$ and is continuous on all of $\mathbb{R}$ because it is a polynomial. In particular, f is continuous on the interval $[2, 6]$. It is clear from the graph of f that there is no number $c \in (2, 6)$ such that $f(c) = 2$. Therefore there is no number $c \in (2, 6)$ such that $1/2 = g(c) = 1/f(c)$.



4. Superman: Escape from Krypton is a vertical roller coaster at the Six Flags Magic Mountain amusement park just outside the Los Angeles area. The coaster has a roughly L-shape track, except of course the part between the horizontal and the vertical parts is rounded, not sharp like the vertex in the letter L. The train takes off horizontally, accelerates, then quickly turns vertical and shoots up while decelerating under the force of gravity. It seems to stop for a split moment at the top of the track before falling back down and eventually returning to the station.
Suppose that

$$h(t) = -5t^2 + 45t + 25$$

is the height of the train at time t during the vertical phase of the ride, where h is measured in meters and t is in seconds.

(a) (9 pts) Use the definition of the derivative (either form) to find the (vertical) velocity of the train at $t = a$.

$$
\begin{aligned}
h'(t) &= \lim_{t\to a} \frac{h(t) - h(a)}{t-a} \\
&= \lim_{t\to a} \frac{-5t^2 + 45t + 25 - (-5a^2 + 45a + 25)}{t-a} \\
&= \lim_{t\to a} \frac{-5t^2 + 45t + 25 + 5a^2 - 45a - 25}{t-a} \\
&= \lim_{t\to a} \frac{-5(t^2 - a^2) + 45(t-a)}{t-a} \\
&= \lim_{t\to a} \frac{-5(t-a)(t+a) + 45(t-a)}{t-a} \\
&= \lim_{t\to a} (-5(t+a) + 45) \\
&= \lim_{t\to a} (-5t - 5a + 45) \\
&= -5a - 5a + 45 \\
&= 45 - 10a
\end{aligned}
$$

where we were able to cancel $t - a$ since $t \neq a$ as $t \to a$, and we evaluated the limit by direct substitution since $-5t - 5a + 45$ is a polynomial in t. Hence the velocity of the train at time $t = a$ is $45 - 10a$ meters/second.

Alternately, we can get the same result by using the other form of the derivative (using $k \to 0$ instead of the usual $h \to 0$ because the name of the function is h):

$$
\begin{aligned}
h'(t) &= \lim_{k\to 0} \frac{h(a+k) - h(a)}{k} \\
&= \lim_{k\to 0} \frac{-5(a+k)^2 + 45(a+k) + 25 - (-5a^2 + 45a + 25)}{k} \\
&= \lim_{k\to 0} \frac{-5(a^2 + 2ak + k^2) + 45(a+k) + 25 + 5a^2 - 45a - 25}{k} \\
&= \lim_{k\to 0} \frac{-5a^2 - 10ak - 5k^2 + 45a + 45k + 25 + 5a^2 - 45a - 25}{k} \\
&= \lim_{k\to 0} \frac{-10ak - 5k^2 + 45k}{k} \\
&= \lim_{k\to 0} (-10a - 5k + 45) \\
&= -10a + 45
\end{aligned}
$$

where we were able to cancel k since $k \neq 0$ as $k \to 0$, and we evaluated the limit by direct substitution since $-10a - 5k + 45$ is a polynomial in t.

(b) (1 pt) Just for fun, how fast is the train moving when it starts it ascent at $t = 0$? Does this seem fast to you?

It would $h'(0) = 45\,\text{m/s}$, which is $3.6(45) = 162\,\text{km/h}$ or about 101 mph. That is fast in an open-air vehicle.

5. (10 pts) **Extra credit problem.** The goal of this problem is to find the derivative of $f(x) = \sqrt[4]{x}$ at $a > 0$. In class, we used the definition of the derivative to find the derivative of $f(x) = \sqrt{x}$ at $a > 0$ by expressing it as

$$f'(a) = \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a},$$

then multiplying the numerator and the denominator by the conjugate $\sqrt{x} + \sqrt{a}$.

We now want to use the definition of the derivative to find the derivative of $f(x) = \sqrt[4]{x}$ at $a > 0$. We can write

$$f'(a) = \lim_{x \to a} \frac{\sqrt[4]{x} - \sqrt[4]{a}}{x - a}.$$

(a) (2 pts) Try using the same trick: multiply the numerator and the denominator by $\sqrt[4]{x} + \sqrt[4]{a}$ and simplify as much as possible.

First, note that $a > 0$ so $\sqrt[4]{a} > 0$. And if x is close enough to a then x is also positive and so is $\sqrt[4]{x}$. Therefore $\sqrt[4]{x} + \sqrt[4]{a}$ is also positive and definitely not 0. Hence we can multiply the numerator and the denominator by it without changing the value of the fraction.

$$\begin{aligned}
\frac{\sqrt[4]{x} - \sqrt[4]{a}}{x - a} &= \frac{(\sqrt[4]{x} - \sqrt[4]{a})(\sqrt[4]{x} + \sqrt[4]{a})}{(x - a)(\sqrt[4]{x} + \sqrt[4]{a})} \\
&= \frac{\sqrt[4]{x}^2 - \sqrt[4]{a}^2}{(x - a)(\sqrt[4]{x} + \sqrt[4]{a})} \\
&= \frac{\sqrt{x} - \sqrt{a}}{(x - a)(\sqrt[4]{x} + \sqrt[4]{a})}.
\end{aligned}$$

(b) (2 pts) What you did in part (a) is not quite enough to evaluate the limit. But it should have resulted in an expression that looks familiar. Try the same trick again: multiply the numerator and the denominator by the conjugate and simplify as much as possible.

The numerator is now $\sqrt{x} - \sqrt{a}$, so we will want to multiply by its conjugate $\sqrt{x} + \sqrt{a}$. Since a is positive and so is x if it is close enough to a, both $\sqrt{x}$ and $\sqrt{a}$ are positive. Therefore $\sqrt{x} + \sqrt{a}$ is also positive and definitely not 0. Hence we can multiply the numerator and the denominator by it without changing the value of the fraction.

$$\begin{aligned}
\frac{\sqrt{x} - \sqrt{a}}{(x - a)(\sqrt[4]{x} + \sqrt[4]{a})} &= \frac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{(x - a)(\sqrt[4]{x} + \sqrt[4]{a})(\sqrt{x} + \sqrt{a})} \\
&= \frac{\sqrt{x}^2 - \sqrt{a}^2}{(x - a)(\sqrt[4]{x} + \sqrt[4]{a})(\sqrt{x} + \sqrt{a})} \\
&= \frac{x - a}{(x - a)(\sqrt[4]{x} + \sqrt[4]{a})(\sqrt{x} + \sqrt{a})}.
\end{aligned}$$

As $x \to a$, x is close to a but $x \neq a$, so $x - a \neq 0$. Therefore we can cancel it:

$$\frac{x - a}{(x - a)(\sqrt[4]{x} + \sqrt[4]{a})(\sqrt{x} + \sqrt{a})} = \frac{1}{(\sqrt[4]{x} + \sqrt[4]{a})(\sqrt{x} + \sqrt{a})}.$$

(c) (6 pts) Use what you know about limits to evaluate the limit you got in part (b). Make sure you carefully justify the steps of your calculation.

$$\lim_{x\to a} \frac{1}{(\sqrt[4]{x}+\sqrt[4]{a})(\sqrt{x}+\sqrt{a})}$$

$$= \frac{\lim_{x\to a} 1}{\lim_{x\to a}[(\sqrt[4]{x}+\sqrt[4]{a})(\sqrt{x}+\sqrt{a})]} \qquad \text{by LL5}$$

$$= \frac{1}{\big(\lim_{x\to a}(\sqrt[4]{x}+\sqrt[4]{a})\big)\big(\lim_{x\to a}(\sqrt{x}+\sqrt{a})\big)} \qquad \text{by LL7 and LL4}$$

$$= \frac{1}{\big(\lim_{x\to a}\sqrt[4]{x}+\lim_{x\to a}\sqrt[4]{a}\big)\big(\lim_{x\to a}\sqrt{x}+\lim_{x\to a}\sqrt{a}\big)} \qquad \text{by LL1}$$

$$= \frac{1}{\big(\sqrt[4]{a}+\sqrt[4]{a}\big)\big(\sqrt{a}+\sqrt{a}\big)} \qquad \text{by LL10 and LL7}$$

$$= \frac{1}{\big(2\sqrt[4]{a}\big)\big(2\sqrt{a}\big)}$$

$$= \frac{1}{4\sqrt[4]{a}\sqrt{a}}$$

$$= \frac{1}{4a^{3/4}}$$

Hence the derivative of $f(x)=\sqrt[4]{x}$ at $a>0$ is $f'(a)=\frac{1}{4a^{3/4}}$.